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PARTICLE APPROXIMATION OF VLASOV EQUATIONS WITH SINGULAR FORCES: PROPAGATION OF CHAOS

BY Maxime HAURAY AND Pierre-Emmanuel JABIN

ABSTRACT. – We justify the mean field approximation and prove the propagation of chaos for a system of particles interacting with a singular interaction force of the type $1/|x|^\alpha$, with $\alpha < 1$ in dimension $d \geq 3$. We also provide results for forces with singularity up to $\alpha < d-1$ but with a large enough cut-off. This last result thus almost includes the case of Coulombian or gravitational interactions, but it also allows for a very small cut-off when the strength of the singularity α is larger but close to one.

RÉSUMÉ. – Nous montrons la validité de l'approximation par champ moyen et prouvons la propagation du chaos pour un système de particules en interaction par le biais d'une force avec singularité $1/|x|^\alpha$, avec $\alpha < 1$ en dimension $d \geq 3$. Nous traitons également le cas de forces avec troncature et des singularités pouvant aller jusqu'à $\alpha < d-1$. Ce dernier résultat permet presque d'atteindre les cas d'interaction coulombiennes ou gravitationnelles et requiert seulement de très petits paramètres de troncature lorsque la singularité est proche de $\alpha = 1$.

1. Introduction

The N particles system. – The starting point is the classical Newton dynamics for N point-particles. We denote by $X_i \in \mathbb{R}^d$ and $V_i \in \mathbb{R}^d$ the position and velocity of the ith particle. For convenience, we also use the notation $Z_i = (X_i, V_i)$ and $Z = (Z_1, \ldots, Z_n)$. Assuming that particles interact two by two with the interaction force $F(x)$, one finds the classical

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E_N(X_i) = \dfrac{1}{N}\displaystyle\sum_{j \neq i} F(X_i - X_j). \end{cases} \tag{1.1}$$

The (N-dependent) initial conditions Z^0 are given. We use the so-called mean-field scaling which consists in keeping the total mass (or charge) of order 1 thus formally enabling us to pass to the limit: this explains the $1/N$ factor in front of the force terms.

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There are many examples of physical systems following (1.1). The best known example concerns Coulombian or gravitational force $F(x) = -\nabla\Phi(x)$, with $\Phi(x) = C/|x|^{d-2}$ for $d \geq 3$ with $C \in \mathbb{R}^*$, which serves as a guiding example and reference. This system then describes particles (ions or electrons) in a one component plasma for $C > 0$, or gravitational interactions for $C < 0$. In the last case the system under study may be a galaxy, a smaller cluster of stars or much larger clusters of galaxies (and thus particles can be "stars" or even "galaxies").

For the sake of simplicity, we consider here only a basic form for the interaction. However the same techniques would apply to more complex models, for instance with several species (electrons and ions in a plasma), 3-particle (or more) interactions, models where the force also depends on the velocity as in swarming models like Cucker-Smale [9]… Indeed a striking feature of our analysis is that it is valid for a force kernel F not necessarily derived from a potential: In fact it never requires any Hamiltonian structure.

The potential and force used in this article. – Our first result applies to interaction forces that are smooth outside of the origin and "weakly" singular near zero, in the sense that they satisfy

$$(1.2) \qquad (S^\alpha) \qquad \exists\, C > 0, \quad \forall\, x \in \mathbb{R}^d \backslash \{0\}, \quad |F(x)| \leq \frac{C}{|x|^\alpha}, \quad |\nabla F(x)| \leq \frac{C}{|x|^{\alpha+1}},$$

for some $\alpha < 1$.

We refer to this condition as the "weakly" singular case because under this, the potential (when it exists) is continuous and bounded near the origin. It is reasonable to expect that the analysis is simpler in that case than with a singular potential.

The second type of potentials or forces that we are dealing with are more singular, satisfying the (S^α)-condition with $\alpha < d-1$, but with an additional cut-off η near the origin that will depend on N

$$(1.3) \qquad (S^\alpha_m) \qquad \begin{array}{l} \text{i)}\ \ F \text{ satisfies a } (S^\alpha)\text{-condition for some } \alpha < d-1, \\ \text{ii)}\ \forall\, |x| \geq N^{-m},\ F_N(x) = F(x), \\ \text{iii)}\ \forall\, |x| \leq N^{-m},\ |F_N(x)| \leq N^{m\alpha}. \end{array}$$

We will refer to that case as the "strongly" singular case. Remark that the interaction kernel F in fact depends on the number of particles. This might seem strange from the physical point of view but it is in fact very common in numerical simulations in order to regularize the interactions.

As we shall see in more details later, we can choose the cut-off parameter smaller than typical inter particle distance (in position) if α is not too large (precisely smaller than $d/2$). In that case one would hope that the cut-off is actually rarely "used".

As the interaction force is singular, we first precise what we mean by solutions to (1.1) in the following definition

Definition 1. – *A (global) solution to* (1.1) *with initial condition*

$$Z^0 = (X^0_1,\ V^0_1, \ldots, X^0_N,\ V^0_N) \in \mathbb{R}^{2dN}$$

(at time 0) is a continuous trajectory $Z(t) = \big(X_1(t),\ V_1(t), \ldots, X_N(t),\ V_N(t)\big)$ *such that*

$$\forall t \in \mathbb{R}^+,\ \forall i \leq N, \qquad \begin{cases} X_i(t) = X_i^0 + \displaystyle\int_0^t V_i(s)\, ds \\ V_i(t) = V_i^0 + \dfrac{1}{N} \displaystyle\sum_{j \neq i} \int_0^t F(X_i(s) - X_j(s))\, ds. \end{cases} \tag{1.4}$$

Local (in time) solutions are defined similarly.

We always assume that such solutions to (1.1) exist, at least for almost all initial configurations of the particles and over any time interval $[0,\ T]$ under consideration. Of course as we use singular interaction forces, this is not completely obvious, but it holds under the assumption (1.2). This point is discussed at the end of the article in Subsection 6.1, and we now focus on the problem raised by the limit $N \to +\infty$.

Remark also that the uniqueness of such solutions is not important for our study. Only the uniqueness of the solution to the limit equation is crucial for the mean-field limit and the propagation of chaos.

The Jeans-Vlasov equation. – At first glance, the system (1.1) might seem quite reasonable. However many problems arise when one tries to use it for practical applications. In our case, the main issue is the number of particles, i.e., the dimension of the system. For example a plasma or a galaxy usually contains a very large number of "particles", typically from 10^9 to 10^{25}, which can make solving (1.1) numerically prohibitively costly.

As usual in this kind of situation, one would like to replace the discrete system (1.1) by a "continuous" model. In our case this model is posed in the space $\mathbb{R}^{2d}$, i.e., it involves the distribution function $f(t, x, v)$ in time, position and velocity. The evolution of that function $f(t, x, v)$ is given by the Jeans-Vlasov equation (or collisionless Boltzmann equation)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0\,, \\ E(x) = \displaystyle\int_{\mathbb{R}^d} \rho(t, y)\, F(x - y)\, dy, \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^d} f(t, x, v)\, dv, \end{cases} \tag{1.5}$$

where here ρ is the spatial density and the initial density f^0 is given.

Our purpose in this article is to understand when and in which sense, Equation (1.5) can be seen as a limit of system (1.1). This question is of importance for theoretical reasons, to justify the validity of the Vlasov equation for example. It also plays a role for numerical simulation in plasma physics [7, 30] and astrophysics [1], where a large class of methods (among which the "Particles in Cells" method) introduce a large number of "virtual" particles (roughly around 10^6 or 10^8, to compare with the real order mentioned above) in order to obtain a many particle system solvable numerically. The problem in that case is to explain why it is possible to correctly approximate the system by using much fewer particles. This would of course be ensured by the convergence of (1.1) to (1.5).

We make use of uniqueness results for the solution to Equation (1.5). The regularity theory for this equation is now well understood, even when the interaction F is singular, including the Coulombian case. The existence of weak solutions goes back to [3, 20]. Existence and uniqueness of global classical solutions in dimension up to 3 is proved in [51], [55]