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BOUNDARY VALUE PROBLEMS
FOR DEGENERATE ELLIPTIC EQUATIONS
AND SYSTEMS

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ABSTRACT. – We study boundary value problems for degenerate elliptic equations and systems with square integrable boundary data. We can allow for degeneracies in the form of an A_2 weight. We obtain representations and boundary traces for solutions in appropriate classes, perturbation results for solvability and solvability in some situations. The technology of earlier works of the first two authors can be adapted to the weighted setting once the needed quadratic estimate is established and we even improve some results in the unweighted setting. The proof of this quadratic estimate does not follow from earlier results on the topic and is the core of the article.

RÉSUMÉ. – On étudie les problèmes aux limites pour les équations et systèmes elliptiques dégé-nérés avec données au bord de carré intégrable. La dégénérescence est contrôlée par un poids A_2. On obtient représentation et trace pour les solutions dans des classes appropriées, des résultats de pertur-bation pour la résolubilité, et résolubilité pour certaines situations. La méthode de travaux antérieurs par les deux premiers auteurs est adaptée à ces dégénérescences une fois démontrée une estimation qua-dratique et nous améliorons même certains résultats dans le cas non-dégénéré. La preuve de cette es-timation quadratique ne se déduit pas en revanche de résultats antérieurs et est la partie centrale de l'article.

1. Introduction

In the series of articles [21, 19, 20], degenerate elliptic equations in divergence form with real symmetric coefficients are studied. There, the degeneracy is given in terms of an A_2 weight or a power of the Jacobian of a quasi-conformal map. The first article gives interior estimates, the second article deals with the Wiener test and the third one study boundary behavior and harmonic measure. Further work along these lines, for exam-ple [22, 13], has been done. However, little work on the fundamental L_p Dirichlet and Neumann problems in the degenerate setting seems to have been done. Here, we want to initiate such study of boundary value problems, with L_2 boundary data, for a large class of weights, in the case of domains which are Lipschitz diffeomorphic to the upper half space

$\mathbb{R}^{1+n}_+ := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\}$, $n \geq 1$. Thus, our work includes the case of special Lipschitz domains. Another difference to earlier work that we want to stress is that we consider general elliptic divergence form systems, and not only scalar equations, as this is the natural setting for the methods used. In this generality, interior pointwise regularity estimates fail in general, even in the uniform elliptic case. However, we emphasize that the methods used in this paper do not require such pointwise estimates.

We consider divergence form second order, real and complex, *degenerate* elliptic systems

$$\text{(1)} \qquad \sum_{i,j=0}^{n} \sum_{\beta=1}^{m} \partial_i \left(A_{i,j}^{\alpha,\beta}(t,x) \partial_j u^\beta(t,x) \right) = 0, \ \alpha = 1, \dots, m$$

in $\mathbb{R}^{1+n}_+$, where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$, which we abbreviate as $\text{div} A \nabla u = 0$, where

$$\text{(2)} \qquad A = (A_{i,j}^{\alpha,\beta}(t,x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m}.$$

We assume A to be degenerate in the sense that for some $w \in A_2(\mathbb{R}^n)$ and $C < \infty$,

$$\text{(3)} \qquad |A(t,x)| \leq C w(x), \ \text{for a.e. } (t,x) \in \mathbb{R}^{1+n}_+$$

and elliptic degenerate in the sense that $w^{-1}A$ is accretive on a space $\mathscr{H}^0$ that we define below. This ellipticity condition means that there exists $\kappa > 0$ such that

$$\text{(4)} \qquad \text{Re} \int_{\mathbb{R}^n} (Af(x), f(x)) dx \geq \kappa \sum_{i=0}^{n} \sum_{\alpha=1}^{m} \int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 w(x) \, dx,$$

for all $f \in \mathscr{H}^0$ and a.e. $t > 0$. We have set

$$(A\xi, \xi) = \sum_{i,j=0}^{n} \sum_{\alpha,\beta=1}^{m} A_{i,j}^{\alpha,\beta}(t,x) \xi_j^\beta \, \overline{\xi_i^\alpha}.$$

The space $\mathscr{H}^0$ is the closed subspace of $L^2(\mathbb{R}^n, w; \mathbb{C}^{m(1+n)})$ consisting of those functions with $\text{curl}_x (f_i^\alpha)_{i=1,\dots,n} = 0$ for all α. The case of equations is when $m = 1$ or, equivalently, when $A_{i,j}^{\alpha,\beta} = A_{i,j} \delta_{\alpha,\beta}$. In this case, the accretivity condition becomes the usual pointwise accretivity

$$\text{(5)} \qquad \text{Re} \sum_{i,j=0}^{n} A_{i,j} \xi_j \overline{\xi_i} \geq \kappa \sum_{i=0}^{n} |\xi_i|^2 w(x),$$

for all $\xi \in \mathbb{C}^{1+n}$ and a.e. $(t,x) \in \mathbb{R}^{1+n}_+$. Observe that the function $(t,x) \mapsto w(x)$ is an A_2 weight in $\mathbb{R}^{1+n}$ if w is an A_2 weight in $\mathbb{R}^n$. So, the degeneracy is a special case of that considered in the works mentioned above. However, for the boundary value problems we wish to consider, this seems a natural class. To our knowledge, this has not been considered before.

A natural question is whether weights could depend on both variables or only on the t-variable. Already in the non-degenerate case there are regularity conditions without which the Dirichlet problem is ill-posed. As for the degenerate case, the well known example from [14] when $A = t^{1-2s} I$, $0 < s < 1$, leading to representation of the fractional Laplacian $(-\Delta)^s$ as the Dirichlet to Neumann operator, is of a nature that our theory cannot cover. In fact, our weights need to have a trace at the boundary in some sense. Another

option is to assume the weight depends on the x-variable only and put the t-perturbations in the coefficients. See below in this introduction and Section 4.

The Equation (1) must be properly interpreted. Solutions are taken with u and ∇u locally in $L^2(\mathbb{R}_+^{1+n}, w(x)dxdt)$, and the equation is taken in the sense of distributions. Note that we allow complex coefficients and systems, so most of the theory developed for real symmetric equations does not apply, and even for real coefficients we want to develop methods regardless of regularity theory.

The boundary value problems can be formulated as follows: find weak solutions u with appropriate interior estimates of $\nabla_{t,x} u$ satisfying one of the following three natural boundary conditions.

- The Dirichlet condition $u = \varphi$ on $\mathbb{R}^n$, given the Dirichlet datum $\varphi \in L^2(\mathbb{R}^n, w; \mathbb{C}^m)$.
- The Dirichlet regularity condition $\nabla_x u = \varphi$ on $\mathbb{R}^n$, given the regularity datum $\varphi \in L^2(\mathbb{R}^n, w; \mathbb{C}^{mn})$ satisfying $\operatorname{curl}_x \varphi = 0$. Alternately, φ is the tangential gradient of the Dirichlet datum.
- The Neumann condition $\partial_{\nu_A} u = (A\nabla_{t,x} u, e_0) = \varphi$ on $\mathbb{R}^n$, given $\varphi \in L^2(\mathbb{R}^n, w^{-1}; \mathbb{C}^m)$. Here e_0 is the upward unit vector in the t-direction.

Observe that the natural space for the gradient at the boundary is $L^2(\mathbb{R}^n, w; \mathbb{C}^{m(1+n)})$, and since A is of the size w, multiplication by A maps into $L^2(w^{-1})$. Thus the weight w^{-1} is natural for the conormal derivative. In order to work with the same weighted space for all three problems, we shall consider the w-normalized conormal derivative $\partial_{\nu_{w^{-1}A}} u|_{t=0} = w^{-1}\partial_{\nu_A} u|_{t=0} \in L^2(\mathbb{R}^n, w; \mathbb{C}^m)$.

Boundary value problems can be formulated for L^p data with $p \neq 2$. This is for later work and here we restrict our attention to $p = 2$. We mention that there are also "intermediate" boundary value problems for regularity/Neumann data in some negative Sobolev spaces based on $L^2(w)$ using fractional powers of the Laplacian $-\Delta_w$ (defined later) that can be treated by the same methods. One important case is the treatment of variational solutions in this context, i.e., those with $\iint_{\mathbb{R}_+^{1+n}} |\nabla u|^2 \, dw(x)dt < \infty$, in which these problems are always well-posed. In the case $w = 1$, the methods have been worked out completely in [34].

Let us also comment on the corresponding degenerate inhomogeneous problem $\operatorname{div} A\nabla u = f$, with $\operatorname{div} A\nabla u$ denoting the left hand side in (1). A study of such equations would be of interest in its own right, but could also prove useful in the study of boundary value problems of the type we study here. Such applications were implemented in [16, 17] in the non-degenerate setting where the coefficients of the operator were assumed to satisfy a Carleson measure condition in place of t-independence. In [16] a duality argument reduced the desired estimate for solutions to the homogeneous equation to an estimate on a solution to an inhomogeneous equation, which could subsequently be proved. While such investigations in the degenerate setting would certainly make the theory more complete, studying the inhomogeneous equation is not our goal here.

We shall obtain a priori representations of solutions and existence of boundary traces (in the almost everywhere sense) for appropriate solution spaces together with various estimates involving non-tangential maximal functions, a characterization of well-posedness, a duality principle between regularity and Dirichlet problems, perturbation results for well-posedness