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Kari ASTALA & Daniel FARACO & Keith M. ROGERS

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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UNBOUNDED POTENTIAL RECOVERY IN THE PLANE

BY KARI ASTALA, DANIEL FARACO AND KEITH M. ROGERS

Dedicated to the memory of Tuulikki

ABSTRACT. – We reconstruct compactly supported potentials with only half a derivative in L^2 from the scattering amplitude at a fixed energy. For this we draw a connection between the recently introduced method of Bukhgeim, which uniquely determined the potential from the Dirichlet-to-Neumann map, and a question of Carleson regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. We also provide examples of compactly supported potentials, with s derivatives in L^2 for any $s < 1/2$, which cannot be recovered by these means. Thus the recovery method has a different threshold in terms of regularity than the corresponding uniqueness result.

RÉSUMÉ. – Nous reconstruisons des potentiels à support compact avec une demi-derivée dans L^2 à partir de l'amplitude de diffusion à énergie fixe. Pour cela, nous établissons un lien entre une méthode récemment introduite par Bukhgeim pour déterminer de façon unique le potentiel à partir de l'application Dirichlet-to-Neumann, et une question de Carleson qui concerne la convergence vers la donnée initiale des solutions de l'équation de Schrödinger dépendante du temps. Nous fournissons également des exemples de potentiels à support compact, avec s dérivées dans L^2 pour tout $s < 1/2$, qui ne peuvent pas être reconstruits par cette méthode. Ainsi, la méthode de reconstruction a un seuil en termes de la régularité qui diffère du résultat d'unicité.

1. Introduction

We consider the Schrödinger equation $\Delta u = Vu$ on a bounded domain Ω in the plane. For each solution u, we are given the value of both u and $\nabla u \cdot n$ on the boundary $\partial\Omega$, where n is the exterior unit normal on $\partial\Omega$. The goal is then to recover the potential V from this information.

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We suppose throughout that $V \in L^2$ is supported on Ω and that 0 is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + V$. Then for each $f \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} \Delta u = Vu \\ u\big|_{\partial\Omega} = f, \end{cases} \tag{1}$$

and the Dirichlet-to-Neumann (DN) map Λ_V can be formally defined by

$$\Lambda_V \,:\, f \mapsto \nabla u \cdot n|_{\partial\Omega}.$$

Then a restatement of our goal is to recover V from knowledge of Λ_V.

We come to this problem via a question of Calderón regarding impedance tomography [14], where f is the electric potential and $\nabla u \cdot n$ is the boundary current, however the DN map $\Lambda_{V-\kappa^2}$ and the scattering amplitude at energy κ^2 are uniquely determined by each other, and indeed the DN map can be recovered from the scattering amplitude (see the appendix for explicit formulae). Thus we are also addressing the question of whether it is possible to recover a potential from the scattering data at a fixed positive energy.

In higher dimensions, Sylvester and Uhlmann proved that smooth potentials are uniquely determined by the DN map [56] (see [43, 44, 16] for nonsmooth potentials and [11, 46, 29] for the conductivity problem). The uniqueness result was extended to a reconstruction procedure by Nachman [38, 39]. The planar case is quite different mathematically as it is not overdetermined. Here the first uniqueness and reconstruction algorithm was proved by Nachman [40] via $\overline{\partial}$-methods for potentials of conductivity type (see also [12] for uniqueness with less regularity). Sun and Uhlmann [52, 54] proved uniqueness for potentials satisfying nearness conditions to each other. Isakov and Nachman [31] then reconstructed the real valued L^p-potentials, $p > 1$, in the case that their eigenvalues are strictly positive. The $\overline{\partial}$-method in combination with the theory of quasiconformal maps gave the uniqueness result for the conductivity equation with measurable coefficients [3]. The problem for the general Schrödinger equation was solved only in 2008 by Bukhgeim [13] for C^1-potentials. Bukhgeim's result has since been improved and extended to treat related inverse problems (see for example [8, 9, 26, 27, 28, 45, 30]).

The aim of this article is to emphasize a surprising connection between the pioneering work of Bukhgeim [13] and Carleson's question [15] regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. Elaborating on this new point of view we obtain a reconstruction theorem for general planar potentials with only half a derivative in L^2, which is sharp with respect to the regularity. The precise statements are given in the forthcoming Corollary 1.3 and Theorem 1.4.

To describe the results in more detail, we recall that the starting point in [13] was to consider solutions to $\Delta u = Vu$ of the form $u = e^{i\psi}(1+w)$, where from now on

$$\psi(z) \equiv \psi_{k,x}(z) = \tfrac{k}{8}(z-x)^2, \qquad z \in \mathbb{C}, \;\; x \in \Omega.$$

Solutions of this type have a long history (see for example [22, 56, 34, 21]), and in this form they were considered first by Bukhgeim. We will recover the potential by measuring a countable number of times on the boundary, so we take $k \in \mathbb{N}$. We will require the homogeneous Sobolev spaces with norm given by $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2}$, where $(-\Delta)^{s/2}$

is defined via the Fourier transform as usual. In Section 3.2, we prove that if the potential V is contained in $\dot{H}^s$ with $0 < s < 1$, and k is sufficiently large, then we can take $w \equiv w_{k,x} \in \dot{H}^s$ with a bound for the norms which is decreasing to zero in k. We write $u_{k,x} = e^{i\psi}\big(1+w\big)$ for these $w \in \dot{H}^s$.

The definition of the DN map, which maps into the dual of $H^{1/2}(\partial\Omega)$ (see the appendix), yields the basic integral formula in inverse problems; Alessandrini's identity. Indeed, if $u, v \in H^1(\Omega)$ satisfy $\Delta u = Vu$ and $\Delta v = 0$, then the formula states that

$$\Big\langle (\Lambda_V - \Lambda_0)\Big[u\big|_{\partial\Omega}\Big], v\big|_{\partial\Omega}\Big\rangle = \int_\Omega V u\, v.$$

Taking $u = u_{k,x}$, which is also in $H^1(\Omega)$, and $v = e^{i\overline{\psi}}$ this yields

$$\Big\langle (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}}\,\Big\rangle = \int_\Omega e^{i(\psi+\overline{\psi})} V(1+w)\,, \tag{2}$$

and so the integral over Ω can be obtained from information on the boundary.

The bulk of the article is concerned with recovering the potential from the integral on the right-hand side of (2). However, in order to calculate the value of the integral, without knowing the value of the potential V inside Ω, we need to calculate the value of the left-hand side of (2). That is to say, we must determine the values of $u_{k,x}$ on the boundary from the DN map. In the case of linear phase, this was achieved by Nachman [40] for L^p-potentials V, with $p > 1$, and Lipschitz boundary (at least for potentials of conductivity type). For C^1-potentials, with C^2-boundary, the result was extended by Novikov and Santacesaria to quadratic phases [45]. Here we show that for quadratic phases almost no regularity is needed. We consider potentials in the inhomogeneous L^2-Sobolev space H^s, defined as before with $(-\Delta)^{s/2}$ replaced by $(\mathrm{I} - \Delta)^{s/2}$. Our starting point is similar to [40] but we give a shorter argument, avoiding single layer potentials.

Theorem 1.1. – *Let $V \in H^s$ with $s > 0$ and suppose that Ω is Lipschitz. Then, for sufficiently large k, we can identify compact operators $\Gamma_{k,x} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, depending on $\Lambda_V - \Lambda_0$, such that*

$$u_{k,x}|_{\partial\Omega} = (\mathrm{I} - \Gamma_{k,x})^{-1}\big[e^{i\psi}|_{\partial\Omega}\big].$$

For C^1-potentials, Bukhgeim [13] proved that the right-hand side of (2), multiplied by $(4\pi)^{-1}k$, converges to $V(x)$ for all $x \in \Omega$, when k tends to infinity. In Section 4, we obtain this convergence for potentials in H^s with $s > 1$. For discontinuous potentials we are no longer able to recover at each point. Instead we bound the fractal dimension of the sets where the recovery fails. As Sobolev spaces are only defined modulo sets of zero Lebesgue measure, we consider first the potential spaces $L^{s,2} = (-\Delta)^{-s/2}L^2(\mathbb{R}^2)$, and bound the Hausdorff dimension of the points where the recovery fails.

Theorem 1.2. – *Let $V \in L^{s,2}$ with $1/2 \leqslant s < 1$. Then*

$$\dim_H \Big\{ x \in \Omega \,:\, \tfrac{k}{4\pi}\Big\langle (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\overline{\psi}}\,\Big\rangle \not\to V(x) \ \ as\ \ k \to \infty \Big\} \leqslant 2 - s.$$