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RIGHT-ANGLED BILLIARDS AND VOLUMES OF MODULI SPACES OF QUADRATIC DIFFERENTIALS ON $\mathbb{C}P^1$

BY Jayadev S. ATHREYA, Alex ESKIN AND Anton ZORICH
WITH AN APPENDIX BY Jon CHAIKA

Abstract. – We use the relation between the volumes of the strata of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}P^1$ and counting functions of the number of (bands of) simple closed geodesics in associated flat metrics with singularities to prove a very explicit formula for the volume of each such stratum conjectured by M. Kontsevich a decade ago.

Applying ergodic techniques to the Teichmüller geodesic flow we obtain quadratic asymptotics for the number of (bands of) closed trajectories and for the number of generalized diagonals in almost all right-angled billiards.

Résumé. – Nous utilisons le lien entre les volumes des strates de différentielles méromorphes quadratiques avec des pôles simples sur $\mathbb{C}P^1$ et les fonctions de comptage du nombre de (cylindres de) géodésiques fermées simples pour la métrique plate associée afin de démontrer une formule très explicite pour le volume des strates, conjecturée par M. Kontsevich il y a une décennie.

En appliquant des techniques ergodiques au flot géodésique de Teichmüller nous obtenons une asymptotique quadratique pour le nombre de (bandes de) trajectoires fermées et le nombre de diagonales généralisées dans presque tout billard à angles « droits ».

1. Introduction

Motivated by the study of computing asymptotics for the number of generalized diagonals and for the number of closed billiard trajectories in right-angled polygons, we were naturally led to questions on Masur-Veech volumes of strata of moduli spaces of quadratic differentials on $\mathbb{C}P^1$. Our main result, explicitly computing these volumes, resolves a conjecture of M. Kontsevich.

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1.1. Volumes of moduli spaces of quadratic differentials

THEOREM 1.1 (Kontsevich Conjecture). – *The volume of any stratum $\mathcal{Q}_1(d_1,\dots,d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{CP}^1$ (i.e., $d_i \in \{-1\,;\,0\}\cup\mathbb{N}$ for $i=1,\dots,k$, and $\sum_{i=1}^k d_i=-4$) is equal to*

$$\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)=2\pi^2\cdot\prod_{i=1}^k v(d_i) \tag{1.1}$$

(where all the zeroes and poles are "named").

Here, the function v is defined on integers n greater than or equal to -1 by

$$v(n):=\frac{n!!}{(n+1)!!}\cdot\pi^n\cdot\begin{cases}\pi & \text{when } n \text{ is odd}\\ 2 & \text{when } n \text{ is even}\end{cases} \tag{1.2}$$

for $n=-1,0,1,2,3\dots$, and the double factorial $n!!=n\cdot(n-2)\cdots$ is the product of all even (respectively odd) positive integers smaller than or equal to n. By convention we set

$$(-1)!!=0!!=1,$$

which implies that

$$v(-1)=1 \qquad \text{and} \qquad v(0)=2.$$

This formula for the volume (up to some normalization factor) was conjectured by M. Kontsevich about ten years ago. It is much simpler than the formula for the volumes of the strata of Abelian differentials found by A. Eskin and A. Okounkov [25].

When this paper was written, there was not a single stratum of quadratic differentials for which the explicit volume was known, though an algorithm of computation was presented in [26]. In addition to this work, there is some very recent progress in evaluation of volumes of low-dimensional strata in genera different from 0. Rigorous formal methods used in [29] (in particular, implementation of the algorithm [26]) are confirmed by independent numerical experiments [12]. However, any known approach involves significant computer-assisted computations, and is limited to volumes of strata of sufficiently small dimension, while Theorem 1.1 provides a simple formula for *all* strata in genus 0.

Returning to our original motivation, we obtain as an important application of Theorem 1.1 asymptotics for the number of closed trajectories and for the number of generalized diagonals in right-angled polygons (see §1.3 below). This choice is particularly natural in the context of this paper since we have to solve an analogous problem for quadratic differentials and to compute the corresponding Siegel-Veech constants $c_{\mathcal{C}}$ for the strata of quadratic differentials in genus 0 anyway: it makes part of the proof of Theorem 1.1. This theorem also immediately provides asymptotics for certain Hurwitz numbers, see §1.2. Another example of applications is discussed in [14] where the values of volumes and the related Siegel-Veech constants are used to compute Lyapunov exponents of the Hodge bundle over hyperelliptic loci in the strata of quadratic differentials and to compute the diffusion rate for interesting families of generalized wind-tree billiards [13].

Strategy of the proof

We start by solving the counting problems for quadratic differentials. The Siegel-Veech constant c_{area} responsible for the exact quadratic asymptotics of the weighted number of bands of regular closed geodesics on almost any flat sphere in a given stratum $\mathcal{Q}(d_1,\dots,d_n)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ was recently computed in [16],

$$c_{\mathrm{area}}(\mathcal{Q}(d_1,\dots,d_n)) = -\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}. \tag{1.3}$$

Developing techniques elaborated in [21] for the strata of Abelian differentials and using the further results from [8] and [39] on the *principal boundary* of the strata of quadratic differentials we express the Siegel-Veech constant c_{area} in genus 0 in terms of the ratio of the volumes of appropriate strata,

$$c_{\mathrm{area}}(\mathcal{Q}(d_1,\dots,d_n)) = \frac{\text{explicit polynomial in volumes of simpler strata}}{\mathrm{Vol}(\mathcal{Q}(d_1,\dots,d_n))}. \tag{1.4}$$

In this way we obtain a series of identities on the volumes of the strata of meromorphic quadratic differentials with at most simple poles in genus zero. The resulting identities recursively determine the volumes of all strata. The proof of Theorem 1.1, given in §5, consists in verifying that the expression (1.1) for the volume satisfies the combinatorial identities implied by (1.3) and (1.4). Part of this verification is performed in Appendix A.

Remark 1.2 (Normalization conventions). – Note that the convention that all zeroes and poles are "named" affects the normalization: we compute the volumes of the corresponding covers over strata with "anonymous" singularities. For example, the stratum $\mathcal{Q}(1,-1^5)$ of quadratic differentials with "anonymous" zeroes and poles is isomorphic to the stratum $\mathcal{H}(2)$ of holomorphic Abelian differentials; by convention the volume elements are chosen to be invariant under this isomorphism. However, by (1.1) we have

$$\mathrm{Vol}\,\mathcal{Q}_1(1,-1^5) = 2\pi^2\cdot v(1)\cdot(v(-1))^5 = 2\pi^2\cdot\frac{\pi^2}{2}\cdot 1^5 = 5!\cdot\frac{\pi^4}{120} = 5!\cdot\mathrm{Vol}\,\mathcal{H}_1(2),$$

which corresponds to 5! ways to *give names* to five simple poles.

Similarly,

$$\mathrm{Vol}\,\mathcal{Q}_1(2,-1^6) = 2\pi^2\cdot v(2)\cdot(v(-1))^6 = 2\pi^2\cdot\frac{4\pi^2}{3}\cdot 1^6 = \frac{6!}{2!}\cdot\frac{\pi^4}{135} = \frac{6!}{2!}\cdot\mathrm{Vol}\,\mathcal{H}_1(1,1).$$

This time there is an extra factor $\frac{1}{2!}$ responsible for *forgetting the names* of the two zeroes of $\mathcal{H}(1,1)$.

1.2. Counting pillowcase covers

One of the ways to compute the volumes of the strata of Abelian or quadratic differentials (actually, the only one before the current paper) is to count *square-tiled surfaces* or *pillowcase covers*, see [25], [26], [27], [51]. In the current paper we proceed in the other direction: we first compute volumes of the strata by an alternative method, and then, as a corollary of volume computation, we get an explicit expression for the leading term of the function counting associated pillowcase covers, when the degree of the cover tends to infinity.