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# A PROOF OF THE LANDAU-GINZBURG/ CALABI-YAU CORRESPONDENCE VIA THE CREPANT TRANSFORMATION CONJECTURE

BY YUAN-PIN LEE, NATHAN PRIDDIS AND MARK SHOEMAKER

ABSTRACT. – We establish a new relationship (the MLK correspondence) between twisted FJRW theory and local Gromov-Witten theory in all genera. As a consequence, we show that the Landau-Ginzburg/Calabi-Yau correspondence is implied by the crepant transformation conjecture for Fermat type in genus zero. We use this to then prove the Landau-Ginzburg/Calabi-Yau correspondence for Fermat type, generalizing the results of A. Chiodo and Y. Ruan in [7].

RÉSUMÉ. – Nous établissons une nouvelle relation (la correspondance MLK) entre la théorie FJRW twistée et la théorie de Gromov-Witten en tout genre. Cela nous permet de montrer que la conjecture de la transformation crépante pour le type de Fermat en genre zéro implique la correspondance de Landau-Ginzburg/Calabi-Yau. Nous nous servons de ce résultat pour prouver la correspondance de Landau-Ginzburg/Calabi-Yau pour le type de Fermat, généralisant les résultats de A. Chiodo et Y. Ruan de [6].

## 0. Introduction

The crepant transformation conjecture ([17, 18]) describes a relationship between the Gromov-Witten theories of $K$-equivalent varieties in terms of analytic continuation and symplectic transformation. Another conjecture inspired by physics, the Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence ([34, 23, 7]), proposes a similar relationship between the Gromov-Witten theory of a Calabi-Yau variety and the FJRW theory of a singularity. The primary goal of this paper is to relate these two conjectures.

### 0.1. FJRW theory and its relatives

FJRW theory was constructed by Fan, Jarvis and Ruan ([19]) as a "Landau-Ginzburg (LG) $A$ model" to verify a conjecture of Witten ([33]). The construction gives a cohomological field theory defined by a virtual class on a cover of the moduli space of curves. It may be viewed as an analogue of Gromov-Witten theory, yielding invariants of a singularity rather than a smooth variety. Roughly, the input of the theory is an *LG pair* $(Q, G)$ where

$Q$ is a quasi-homogeneous polynomial $Q : \mathbb{C}^N \to \mathbb{C}$, and $G$ an *admissible group* of diagonal automorphisms of $Q$ (See Section 1.1). The moduli space is defined to be $N$-tuples of line bundles on curves, $\mathcal{L}_i \to \mathcal{C}$, such that for each monomial $Q_s = Q_s(x_1, \ldots, x_N)$ in $Q$, $Q_s(\mathcal{L}_1, ..., \mathcal{L}_N) \cong \omega_{\mathcal{C},\log}$, the log-canonical bundle. The most difficult part of the construction is to define the virtual classes. This was done in the analytic category by Fan-Jarvis-Ruan in [19] and in the algebraic category by Polishchuk-Vaintrob in [29].

This paper begins with the observation that in the moduli problem for FJRW theory, one may replace the log-canonical bundle with a power of the log-canonical bundle. This yields new collection of moduli spaces, and for any given power of the log-canonical bundle one may construct a corresponding cohomological field theory. For instance, if one takes the zeroth power, i.e., if we consider the trivial line bundle, then one recovers the orbifold GW theory for the abelian quotient stack $[\mathbb{C}^N/G]$.

It turns out that many of the cohomological field theories associated to the various choices of power of $\omega_{\mathcal{C},\log}$ are equivalent in a precise sense. We dub this the "multiple log-canonical" or *MLK correspondence* (Theorem 5.5). The first important example of this correspondence is the following: restrict to LG pairs $(Q, G)$ where $Q$ is a quasi-homogeneous polynomial of Fermat type i.e., $Q = \sum_{i=1}^N x_i^{d/c_i}$. In this case the MLK correspondence has the following interesting implication, that the genus zero FJRW theory of $(Q, G)$ is determined completely (and explicitly) from the genus zero orbifold GW theory of $[\mathbb{C}^N/G]$ (Theorem 5.11).

### 0.2. The crepant transformation conjecture and the LG/CY correspondence

The crepant transformation conjecture (CTC) (see [17, 18]) predicts that the Gromov-Witten invariants of $K$-equivalent varieties should be related. The relationship is complicated: generating functions of the respective invariants are identified via analytic continuation and symplectic transformation by an element Givental's symplectic loop group ([21, 27]). At this point the CTC is considered well-understood, at least in genus zero, and has been established for a wide class of examples (see, e.g., [17, 16, 28]).

The LG/CY correspondence is a similar type of conjecture, this time connecting the Gromov-Witten theory of certain Calabi-Yau varieties to the FJRW theory a related singularity. The first mathematical proof of the LG/CY correspondence was given by Chiodo and Ruan for the quintic threefold ([7]). When $Q$ is the Fermat quintic in five variables and $G = \mathrm{diag}(\mu_5)$, they proved that the genus zero FJRW theory of $(Q, G)$ is equivalent to genus zero Gromov-Witten theory of the quintic hypersurface $Z(Q) = \{Q = 0\}$ in $\mathbb{P}^4$. The identification of the two theories is striking in that it takes exactly the same form as in the CTC, namely, analytic continuation and symplectic transformation.

Via the MLK correspondence, we explain this similarity by proving that a general form of the LG/CY correspondence can be deduced from the CTC. Symbolically we write:

$$\text{CTC} \Rightarrow \text{LG/CY}.$$

More precisely, let $\mathbb{P}(G) := [\mathbb{P}(c_1, \ldots, c_N)/\bar{G}]$, where $\bar{G}$ is the quotient of $G$ by those elements acting trivially on $[\mathbb{P}(c_1, \ldots, c_N)]$. Let $K_{\mathbb{P}(G)}$ denote the total space of the canonical bundle over $\mathbb{P}(G)$ and let $Z(Q) \subset \mathbb{P}(G)$ be the Calabi-Yau orbifold defined by $Q$. Then the

LG/CY correspondence may be established by a special case of the CTC. The relationship is summarized in the following diagram:

(0.2.1)
$$\begin{tikzcd} GW_0(K_{P(G)}) \arrow[r, "\text{QSD}"] \arrow[d, leftrightarrow, "\text{CTC}"] & GW_0(Z(Q)) \arrow[d, leftrightarrow, "\text{LG/CY}"] \\ GW_0([\mathbb{C}^N/G]) \arrow[r, "\text{MLK}"] & \mathrm{FJRW}_0(Q,G). \end{tikzcd}$$

In the upper right corner is the genus zero GW theory for the Calabi-Yau orbifold Z(Q). The lower right corner is the genus zero FJRW theory associated to the LG pair $(Q, G)$. The right vertical arrow is the LG/CY correspondence discussed above. The left vertical arrow is the CTC relating the genus zero orbifold GW theory of $[\mathbb{C}^N/G]$ with the genus zero GW theory of its crepant partial resolution $K_{\mathbb{P}(G)}$. The upper horizontal arrow is quantum Serre duality ([15]), which relates the GW theory of the total space of a line bundle with the GW theory of the hypersurface defined by a section of the line bundle. The MLK correspondence, established in Section 5.4.3, completes the square.

### 0.3. A remark on proofs

The LG/CY correspondence has previously been proven in genus zero in all cases where the Calabi-Yau is a hypersurface in projective space ([6]) as well as for the mirror quintic ([30]). In addition to generalizing the previous results, we hope that the proof described above provides a useful conceptual framework in which to view the LG/CY correspondence.

In the physics literature, the LG/CY correspondence is understood as an example of a *phase transition* ([34]). From this perspective, the origin of the correspondence is quite natural, and completely analogous to the wall crossing phenomenon arising in the CTC (see [34] and [23] for more information). However the mathematical proofs of the LG/CY correspondence to-date have been computational in nature, and do not shed light on why such a correspondence should hold. The present result then, which shows that the LG/CY correspondence in fact follows from the CTC, may be viewed as a mathematical justification for the relationship between these two conjectures, and a verification of the physical insight which predicts such a correspondence.

Beyond simply verifying the physical description of the LG/CY correspondence, the structure in (0.2.1) may be useful for better understanding the relationship in higher genus. Recall that both the CTC and LG/CY correspondence are given in terms of analytic continuation and symplectic transformation by an element of Givental's symplectic loop group ([21, 27]).

Due to the existence of stabilizers in the action of the symplectic loop group, relating the genus zero FJRW and GW theory in this correspondence requires one to make a choice of symplectic transformation. Crucially, two symplectic transformations which have the same effects on the genus zero theory might have quantizations which act differently on higher genus theories. Therefore, any correspondence in higher genus requires a canonical way of choosing the symplectic transformation relating the genus zero invariants.

In the case of the crepant transformation between $K$-equivalent varieties, there is indeed such a canonical choice, coming from the Fourier-Mukai functor in $K$-theory (see [26]). In the case of the LG/CY correspondence however, there is a priori no canonical choice of symplectic transformation. However we may obtain one via diagram (0.2.1). The symplectic