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QUASINEUTRAL LIMIT FOR VLASOV-POISSON WITH PENROSE STABLE DATA

BY DANIEL HAN-KWAN AND FRÉDÉRIC ROUSSET

ABSTRACT. – We study the quasineutral limit of a Vlasov-Poisson system that describes the dynamics of ions in a plasma. We handle data with Sobolev regularity under the sharp assumption that the profiles in velocity of the initial data satisfy a Penrose stability condition.

As a byproduct of our analysis, we obtain a well-posedness theory for the limit equation (which is a Vlasov equation with Dirac measure as interaction kernel), for such data.

RÉSUMÉ. – Nous étudions la limite quasineutre d'un système de Vlasov-Poisson qui décrit la dynamique d'ions dans un plasma. Nous travaillons avec des données à régularité Sobolev sous l'hypothèse optimale que les profils en vitesse des données initiales satisfont une condition de stabilité de Penrose.

Comme corollaire de notre analyse, nous obtenons une théorie d'existence et d'unicité pour l'équation limite (qui est une équation de Vlasov avec une mesure de Dirac pour noyau d'interaction), pour de telles données.

1. Introduction and main results

We study the quasineutral limit, that is the limit $\varepsilon \to 0$, for the following Vlasov-Poisson system describing the dynamics of ions in the presence of massless electrons:

$$\begin{cases} \partial_t f_\varepsilon \,+\, v\cdot\nabla_x f_\varepsilon \,+\, E_\varepsilon\cdot\nabla_v f_\varepsilon \,=\, 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ V_\varepsilon - \varepsilon^2 \Delta V_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\ f_\varepsilon|_{t=0} = f_\varepsilon^0. \end{cases} \tag{1.1}$$

In these equations, the function $f_\varepsilon(t,x,v)$ stands for the distribution functions of the ions in the phase space $\mathbb{T}^d\times\mathbb{R}^d$, $d\in\mathbb{N}^*$, with $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. We assumed that the density of the electrons n_e satisfies a linearized Maxwell-Boltzmann law, that is $n_e = e^{V_\varepsilon} \sim 1+V_\varepsilon$, which accounts for the source $-(1+V_\varepsilon)$ in the Poisson equation. Such a model was recently studied for instance in [19, 20, 21, 12]. Though we have focused on this simplified law, the arguments in this paper could be easily adapted to the model where the potential is given by the Poisson equation $-\varepsilon^2 \Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv - e^{V_\varepsilon}$.

The dimensionless parameter ε is defined by the ratio between the Debye length of the plasma and the typical observation length. It turns out that in most practical situations, ε is very small, so that the limit $\varepsilon \to 0$, which bears the name of quasineutral limit, is relevant from the physical point of view. Observe that in the regime of small ε, we formally have that the density of ions is almost equal to that of electrons, hence the name quasineutral. This regime is so fundamental that it is even sometimes included in the very definition of a plasma, see e.g., [8].

The quasineutral limit for the Vlasov-Poisson system with the Poisson equation

$$-\varepsilon^2 \Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv - \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon \, dv \, dx, \tag{1.2}$$

that describes the dynamics of electrons in a fixed neutralizing background of ions is also very interesting. Nevertheless, we shall focus in this paper on the study of (1.1). The study of (1.2) combines the difficulties already present in this paper linked to kinetic instabilities and those related to high frequency waves due to the large electric field that, do not occur in the case of (1.1). The study of the combination of these two phenomena is postponed to future work.

It is straightforward to obtain the formal quasineutral limit of (1.1) as $\varepsilon \to 0$: we expect that $\varepsilon^2 \Delta V_\varepsilon$ tends to zero and hence if f_ε converges in a reasonable way to some f, then f should solve

$$\begin{cases} \partial_t f \ + \ v \cdot \nabla_x f \ + \ E \cdot \nabla_v f \ = \ 0, \\ E = -\nabla_x \rho, \quad \rho = \displaystyle\int_{\mathbb{R}^d} f \, dv, \\ f|_{t=0} = f^0. \end{cases} \tag{1.3}$$

This system was named *Vlasov-Dirac-Benney* by Bardos [1] and studied in [3, 2]. It was also referred to as the *kinetic Shallow Water system* in [20] by analogy with the classical Shallow Water system of fluid mechanics. In particular, it was shown in [3] that the semigroup of the linearized system around *unstable* equilibria is unbounded in Sobolev spaces (even with loss of derivatives). This yields the ill-posedness of (1.3) in Sobolev spaces, see in particular the recent work [24]. In [2], it was nevertheless shown in dimension one, i.e., for $d = 1$ that (1.3) is well-posed in the class of functions $f(x, v)$ such that for all $x \in \mathbb{T}$, $v \mapsto f(x, v)$ is compactly supported and is increasing for $v \leq m(t, x)$ and then decreasing for $v \geq m(t, x)$, that is to say for functions that for all x have the shape of *one bump*. The method in [2] is to reduce the problem to an infinite number of fluid type equations by using a water bag decomposition.

The mathematical study of the quasineutral limit started in the nineties with pioneering works of Brenier and Grenier for Vlasov with the Poisson Equation (1.2), first with a limit involving defect measures [7, 14], then with a full justification of the quasineutral limit for initial data with uniform analytic regularity [15]. The work [15] also included a description of the so-called plasma waves, which are time oscillations of the electric field of frequency and amplitude $O(\frac{1}{\varepsilon})$. As already said, such oscillations actually do not occur in the quasineutral limit of (1.1). More recently, in [23, 22], relying on Wasserstein stability estimates inspired from [25, 28], it was proved that exponentially small but rough perturbations are allowed in the main result of [15].

In analytic regularity, it turns out that instabilities for the Vlasov-Poisson system, such as two-stream instabilities, do not have any effect, whereas in the class of Sobolev functions, they definitely play a crucial role. It follows that the quasineutral approximation both for (1.1) and (1.2) is not always valid. In particular, the convergence of (1.1) to (1.3) does not hold in general: we refer to [16, 21].

Nevertheless, it can be expected that the formal limit can be justified in Sobolev spaces for stable situations. We shall soon be more explicit about what we mean by stable data, but this should at least be included in the class of data for which the expected limit system (1.3) is well-posed. The first result in this direction is due to Brenier [6] (see also [30] and [20]), in which he justifies the quasineutral limit for initial data converging to a monokinetic distribution, that is a Dirac mass in velocity. This corresponds to a stable though singular case since the Dirac mass can be seen as an extremal case of a Maxwellian, that is a function with one bump. Brenier introduced the so-called modulated energy method to prove this result. Note that in this case the limit system is a fluid system (the incompressible Euler equations in the case of (1.2) or the shallow water equations in the case of (1.1)) and not a kinetic equation. This result is coherent with the fact that the instabilities present at the kinetic level do not show up at the one-fluid level, for example the quasineutral limit of the Euler-Poisson system can be justified in Sobolev spaces as shown for example in [9, 27], among others.

For non singular stable data with Sobolev regularity, there are only few available results which all concern the one-dimensional case $d = 1$.

- In [21], using the modulated energy method, the quasineutral limit is justified for very special initial data namely initial data converging to one bump functions that are furthermore *symmetric* and space homogeneous (thus that are stationary solutions to (1.1) and (1.3)). It is also proved that this is the best we could hope for with this method.
- Grenier sketched in [16] a result of convergence for data such that for every x the profile in v has only one bump. The proposed proof involves a functional taking advantage of the monotonicity in the one bump structure. Such kind of functionals have been recently used in other settings, for example in the study of the hydrostatic Euler equation or the Prandtl equation, see for example [31, 32, 11].

The main goal of this work is to justify the quasineutral limit that is to prove the derivation of (1.3) from (1.1) in the general stable case and in any dimension. As we shall see below a byproduct of the main result is the well-posedness of the system (1.3) in any dimension for smooth data with finite Sobolev regularity such that for every x, the profile $v \mapsto f^0(x,v)$ satisfies a Penrose stability condition. This condition is automatically satisfied in dimension one by smooth functions that for every x have a "one bump" profile, as well as by small perturbations of such functions.

To state our results, we shall first introduce the Penrose stability condition [36, 35] for homogeneous equilibria $\mathbf{f}(v)$. Let us define for the profile $\mathbf{f}$ the Penrose function

$$\mathscr{P}(\gamma,\tau,\eta,\mathbf{f}) = 1-(2\pi)^d\int_0^{+\infty} e^{-(\gamma+i\tau)s}\,\frac{i\eta}{1+|\eta|^2}\cdot(\mathscr{F}_v\nabla_v\mathbf{f})(\eta s)\,ds,$$