ISSN 0012-9593

quatrième série - tome 49 fascicule 1 janvier-février 2016

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Caucher BIRKAR

Existence of flips and minimal models for 3-folds in char p

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 49, 2016, p. 169 à 212

EXISTENCE OF FLIPS AND MINIMAL MODELS FOR 3-FOLDS IN CHAR p

BY CAUCHER BIRKAR

To Tarn and Zanko, with love

ABSTRACT. – We will prove the following results for 3-fold pairs (X, B) over an algebraically closed field k of characteristic $p > 5$: log flips exist for $\mathbb{Q}$-factorial dlt pairs (X, B); log minimal models exist for projective klt pairs (X, B) with pseudo-effective $K_X + B$; the log canonical ring $R(K_X + B)$ is finitely generated for projective klt pairs (X, B) when $K_X + B$ is a big $\mathbb{Q}$-divisor; semi-ampleness holds for a nef and big $\mathbb{Q}$-divisor D if $D - (K_X + B)$ is nef and big and (X, B) is projective klt; $\mathbb{Q}$-factorial dlt models exist for lc pairs (X, B); terminal models exist for klt pairs (X, B); ACC holds for lc thresholds, etc.

RÉSUMÉ. – Étant donnée une paire (X, B) de dimension trois sur un corps algébriquement clos k de caractéristique $p > 5$, nous prouvons les résultats suivants : existence de log-flips lorsque la paire est $\mathbb{Q}$-factorielle et dlt; existence de log-modèles minimaux lorsque la paire est klt, projective, et avec $K_X + B$ pseudo-effectif ; finitude de l'anneau log-canonique $R(K_X + B)$ lorsque la paire est klt, projective, et avec $K_X + B$ gros; semi-amplitude pour un $\mathbb{Q}$-diviseur nef et gros D, sous la condition que $D - (K_X + B)$ est nef et gros et que (X, B) est klt et projective; existence de modèles dlt et $\mathbb{Q}$-factoriels lorsque la paire est lc; existence de modèles terminaux lorsque la paire est klt; validité de la Conjecture ACC pour le seuil lc, etc.

1. Introduction

We work over an algebraically closed field k of characteristic (char) $p > 0$. The pairs (X, B) we consider in this paper always have $\mathbb{R}$-boundaries B unless otherwise stated.

Higher dimensional birational geometry in char p is still largely conjectural. Even the most basic problems such as base point freeness are not solved in general. Ironically though Mori's work on existence of rational curves which plays an important role in characteristic 0 uses reduction mod p techniques. There are two reasons among others which have held back progress in char p: resolution of singularities is not known and Kawamata-Viehweg vanishing fails. However, it was expected that one can work out most components of the minimal model program in dimension 3. This is because resolution of singularities is known in dimension 3 and many problems can be reduced to dimension 2 hence one can use special features of surface geometry.

On the positive side there has been some good progress toward understanding birational geometry in char p. People have tried to replace the characteristic 0 tools that fail in char p. For example, Keel [16] developed techniques for dealing with the base point free problem and semi-ampleness questions in general without relying on Kawamata-Viehweg vanishing type theorems. On the other hand, motivated by questions in commutative algebra, people have introduced Frobenius-singularities whose definition does not require resolution of singularities and they are very similar to singularities in characteristic 0 (cf. [23]).

More recently Hacon-Xu [13] proved the existence of flips in dimension 3 for pairs (X, B) with B having standard coefficients, that is, coefficients in $\mathfrak{S} = \{1 - \frac{1}{n} \mid n \in \mathbb{N} \cup \{\infty\}\}$, and char $p > 5$. From this they could derive existence of minimal models for 3-folds with canonical singularities. In this paper, we rely on their results and ideas. The requirement $p > 5$ has to do with the behavior of singularities on surfaces, e.g., a klt surface singularity over k of char $p > 5$ is strongly F-regular.

Log flips. – Our first result is on the existence of flips.

THEOREM 1.1. – *Let (X, B) be a $\mathbb{Q}$-factorial dlt pair of dimension 3 over k of char $p > 5$. Let $X \to Z$ be a $K_X + B$-negative extremal flipping projective contraction. Then its flip exists.*

The conclusion also holds if (X, B) is klt but not necessarily $\mathbb{Q}$-factorial. This follows from the finite generation below (1.3). The theorem is proved in Section 6 when X is projective. The quasi-projective case is proved in Section 8. We reduce the theorem to the case when X is projective, B has standard coefficients, and some component of $\lfloor B \rfloor$ is negative on the extremal ray: this case is [13, Theorem 4.12] which is one of the main results of that paper. A different approach is taken in [7] to prove 1.1 when B has hyperstandard coefficients and $p \gg 0$ (these coefficients are of the form $\frac{n-1}{n} + \sum \frac{l_i b_i}{n}$ where $n \in \mathbb{N} \cup \{\infty\}$, $l_i \in \mathbb{Z}^{\geq 0}$ and b_i are in some fixed DCC set).

To prove Theorem 1.1 we actually first prove the existence of *generalized flips* [13, after Theorem 5.6]. See Section 6 for more details.

Log minimal models. – In [13, after Theorem 5.6], using generalized flips, a *generalized LMMP* is defined which is used to show the existence of minimal models for varieties with canonical singularities (or for pairs with canonical singularities and "good" boundaries). Using weak Zariski decompositions as in [4], we construct log minimal models for klt pairs in general.

THEOREM 1.2. – *Let (X, B) be a klt pair of dimension 3 over k of char $p > 5$ and let $X \to Z$ be a projective contraction. If $K_X + B$ is pseudo-effective$/Z$, then (X, B) has a log minimal model over Z.*

The theorem is proved in Section 8. Alternatively, one can apply the methods of [3] to construct log minimal models for lc pairs (X, B) such that $K_X + B \equiv M/Z$ for some $M \geq 0$. Note that when $X \to Z$ is a semi-stable fibration over a curve and $B = 0$, the theorem was proved much earlier by Kawamata [14].

Remark on Mori fibre spaces. – Let (X, B) be a projective klt pair of dimension 3 over k of char $p > 5$ such that $K_X + B$ is not pseudo-effective. An important question is whether (X, B) has a Mori fibre space. There is an ample $\mathbb{R}$-divisor $A \geq 0$ such that $K_X + B + A$ is pseudo-effective but $K_X + B + (1 - \epsilon)A$ is not pseudo-effective for any $\epsilon > 0$. Moreover, we may assume that $(X, B + A)$ is klt as well (9.2). By Theorem 1.2, $(X, B + A)$ has a log minimal model $(Y, B_Y + A_Y)$. Since $K_Y + B_Y + A_Y$ is not big, $K_Y + B_Y + A_Y$ is numerically trivial on some covering family of curves by [9] (see also 1.11 below). Again by [9], there is a nef reduction map $Y \dashrightarrow T$ for $K_Y + B_Y + A_Y$ which is projective over the generic point of T. Although $Y \dashrightarrow T$ is not necessarily a Mori fibre space, in some sense it is similar.

Finite generation, base point freeness, and contractions. – We will prove finite generation in the big case from which we can derive base point freeness and contractions of extremal rays in many cases. These are proved in Section 10.

THEOREM 1.3. – *Let (X, B) be a klt pair of dimension 3 over k of char $p > 5$ and $X \to Z$ a projective contraction. Assume that $K_X + B$ is a $\mathbb{Q}$-divisor which is big$/Z$. Then the relative log canonical algebra $\mathcal{R}(K_X + B/Z)$ is finitely generated over $\mathcal{O}_Z$.*

Assume that Z is a point. If K_X+B is not big, then $R(K_X+B/Z)$ is still finitely generated if $\kappa(K_X + B) \leq 1$. It remains to show the finite generation when $\kappa(K_X + B) = 2$: this can probably be reduced to dimension 2 using an appropriate canonical bundle formula, for example as in [9].

A more or less immediate consequence of the above finite generation is the following base point freeness.

THEOREM 1.4. – *Let (X, B) be a projective klt pair of dimension 3 over k of char $p > 5$ and $X \to Z$ a projective contraction where B is a $\mathbb{Q}$-divisor. Assume that D is a $\mathbb{Q}$-divisor such that D and $D - (K_X + B)$ are both nef and big$/Z$. Then D is semi-ample$/Z$.*

Assume that Z is a point. When $D - (K_X + B)$ is nef and big but D is nef with numerical dimension $\nu(D)$ one or two, semi-ampleness of D is proved in [9] under some restrictions on the coefficients.

THEOREM 1.5. – *Let (X, B) be a projective $\mathbb{Q}$-factorial dlt pair of dimension 3 over k of char $p > 5$, and $X \to Z$ a projective contraction. Let R be a $K_X + B$-negative extremal ray$/Z$. Assume that there is a nef and big$/Z$ $\mathbb{Q}$-divisor N such that $N \cdot R = 0$. Then R can be contracted by a projective morphism.*

Note that if $K_X + B$ is pseudo-effective$/Z$, then for every $K_X + B$-negative extremal ray R/Z there exists N as in the theorem (see 3.3). Therefore such extremal rays can be contracted by projective morphisms.

Theorems 1.4 and 1.5 have been proved by Xu [30] independently and more or less at the same time but using a different approach. His proof also relies on our results on flips and minimal models.