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PARITY SHEAVES AND TILTING MODULES

BY Daniel JUTEAU, Carl MAUTNER
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Abstract. – We show that tilting modules and parity sheaves on the affine Grassmannian are related through the geometric Satake correspondence, when the characteristic is bigger than an explicit bound.

Résumé. – Nous montrons que la correspondance de Satake géométrique fait correspondre les modules basculants aux faisceaux à parité sur la grassmannienne affine, lorsque la caractéristique est plus grande qu'une borne explicite.

1. Introduction

1.1. Tilting modules for reductive groups

Let $\mathbf{G}$ be a split reductive group over a field k of characteristic p with a chosen maximal torus and Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$. Let Λ^+ denote the set of dominant weights, where dominance is defined with respect to the system of positive roots which are opposite to the $\mathbf{T}$-weights in the Lie algebra of $\mathbf{B}$. To each dominant weight $\lambda \in \Lambda^+$, is associated an induced representation $\nabla_\lambda := \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}} k_\lambda = H^0(\mathbf{G}/\mathbf{B}, \mathcal{O}(\lambda))$ and its dual Δ_λ, the Weyl module.

The rational representations of $\mathbf{G}$ form a highest weight category in which the Weyl modules are the standard objects and the induced modules are the costandard objects. A rational representation is said to be tilting, if it admits two filtrations—one with successive quotients isomorphic to Weyl modules and the other with successive quotients isomorphic to induced modules.

A theorem of Ringel [30, Proposition 2] about general highest weight categories specializes in this setting to the following result [10, Theorem 1.1],

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THEOREM 1.1. – *For each $\lambda \in \Lambda^+$ (up to non-canonical isomorphism), there exists a unique indecomposable tilting module $T(\lambda)$ which has a unique highest weight λ. Moreover, λ has multiplicity one as a weight of $T(\lambda)$. Every indecomposable tilting module is isomorphic to $T(\lambda)$ for some $\lambda \in \Lambda^+$.*

An interesting feature of the class of tilting modules is that it is closed under both tensor product and restriction to a Levi subgroup:

THEOREM 1.2. – *If T and T' are tilting modules for $\mathbf{G}$, then so is the tensor product $T \otimes T'$.*

THEOREM 1.3. – *Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. If T is a tilting module for $\mathbf{G}$, then the restriction $\mathrm{Res}_{\mathbf{L}}^{\mathbf{G}} T$ to $\mathbf{L}$ is a tilting module for $\mathbf{L}$.*

Results in this direction were originally motivated by a question of Humphreys, who asked if the tensor product of two Weyl modules has a Weyl filtration. Theorem 1.2 was first proven by Wang [32] in type A and in large characteristic for other groups. Donkin [9] later proved both theorems in almost full generality (he excluded the case when $p = 2$ and G has a component of type E_7 or E_8). The first complete and uniform proof of both theorems is due to Mathieu [25] and uses Frobenius splitting techniques. Other approaches to the first theorem appear in [21, 29, 28, 18].(1)

1.2. Parity sheaves for the affine Grassmannian

Let $\check{G} \supset \check{T}$ be the connected complex algebraic group and maximal torus with root datum dual to that of $\mathbf{G}$.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. The affine Grassmannian $\mathcal{G}r$ for $\check{G}$ is an ind-scheme whose complex points form the set $\check{G}(\mathcal{K})/\check{G}(\mathcal{O})$. We consider its complex points as an ind-$\check{G}(\mathcal{O})$-variety. The $\check{G}(\mathcal{O})$-orbits are labeled by the set Λ^+ of dominant weights of $\mathbf{G}$ and we denote the orbit corresponding to a weight λ by $\mathcal{G}r^\lambda$.

The geometric Satake theorem [27] shows that the representation theory of $\mathbf{G}$ is encoded in a category of sheaves of k-vector spaces on $\mathcal{G}r$. More precisely, the category of rational representations of $\mathbf{G}$ is equivalent to the category $P(\mathcal{G}r)$: the category of $\check{G}(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r$ with coefficients in k.

The category $P(\mathcal{G}r)$ is the heart of a t-structure on $D(\mathcal{G}r)$, the bounded $\check{G}(\mathcal{O})$-equivariant constructible derived category of sheaves of k-vector spaces on $\mathcal{G}r$. There is a natural convolution product $\star : D(\mathcal{G}r) \times D(\mathcal{G}r) \to D(\mathcal{G}r)$ which is t-exact and produces a tensor structure on $P(\mathcal{G}r)$, corresponding under the equivalence to the tensor product of rational representations of $\mathbf{G}$.

Similarly, for any Levi subgroup $\mathbf{L} \subset \mathbf{G}$, the restriction functor from $\mathbf{G}$ to $\mathbf{L}$ corresponds to a geometrically-defined t-exact functor $R_{\check{L}}^{\check{G}} : D(\mathcal{G}r) \to D(\mathcal{G}r_{\check{L}})$, where $\mathcal{G}r_{\check{L}}$ is the affine Grassmannian for $\check{L} \subset \check{G}$, the Levi subgroup containing $\check{T}$ whose roots are dual to those of $\mathbf{L}$ (see Section 2.3 for more details).

(1) In the literature, these theorems appear with the words 'tilting modules' replaced by 'modules admitting a good filtration' or 'modules admitting a Weyl filtration'. In the appendix to this paper, we explain the fact, well-known to experts, that these are equivalent formulations.

Recall the notion of a parity complex [17, Section 2.2].[(2)] The affine Grassmannian is a Kac-Moody flag variety and hence the results from [17, Section 4.1] (see in particular Example 4.2 of loc. cit.) can be used to study $D(\mathcal{G}r)$. Parity complexes on the affine Grassmannian behave very much like the tilting modules for $\mathbf{G}$. In particular, we have the following theorems which mirror the ones for tilting modules.

The starting point is a result [17, Theorem 4.6] that is very similar to Theorem 1.1:

THEOREM 1.4. – *Assume that the characteristic of k is not a torsion prime for $\check{G}$.*[(3)]

For each $\lambda \in \Lambda^+$ (up to non-canonical isomorphism), there exists a unique indecomposable parity complex $\mathcal{E}(\lambda)$ such that $\operatorname{supp}(\mathcal{E}(\lambda)) = \overline{\mathcal{G}r^\lambda}$ *and* $\mathcal{E}(\lambda)|_{\mathcal{G}r^\lambda} = \underline{k}_{\mathcal{G}r^\lambda}[\dim \mathcal{G}r^\lambda]$. *Every indecomposable parity complex is isomorphic to $\mathcal{E}(\lambda)[m]$ for some $\lambda \in \Lambda^+$ and $m \in \mathbb{Z}$.*

The indecomposable parity complexes $\mathcal{E}(\lambda)$ are known as parity sheaves.

As a special case of [17, Theorem 4.8], we obtain an analogue of Theorem 1.2:

THEOREM 1.5. – *If $\mathcal{F} \in D(\mathcal{G}r)$ and $\mathcal{G} \in D(\mathcal{G}r)$ are parity complexes, then so is the convolution product $\mathcal{F} \star \mathcal{G} \in D(\mathcal{G}r)$.*

The first part of this paper establishes an analogue of Theorem 1.3. In Section 2, we prove the following result,

THEOREM 1.6. – *Let $\check{L}$ be the Langlands dual of $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$. If $\mathcal{F} \in D(\mathcal{G}r)$ is a parity complex, then $R_{\check{L}}^{\check{G}}(\mathcal{F})$ is a parity complex on the affine Grassmannian for $\check{L}$.*

The idea of the proof is to replace the purity argument of [7, Theorem 2] by a parity argument.

1.3. Tilting equals parity

In Section 3, which can be read independently of Section 2, we prove our main result, which explains the similarities between the theorems stated above. Our result shows that, for most characteristics, the Theorems 1.2 and 1.3 about tilting modules are equivalent to the Theorems 1.5 and 1.6 about parity sheaves.

Recall [27, Prop. 13.1] that, for $\lambda \in \Lambda^+$ a dominant weight, the Weyl module Δ_λ (resp. ∇_λ) goes under the geometric Satake equivalence to the standard sheaf ${}^p\mathcal{J}_!(\lambda) := {}^p j_{\lambda !}\underline{k}_\lambda[d_\lambda]$ (resp. costandard sheaf ${}^p\mathcal{J}_*(\lambda) := {}^p j_{\lambda *}\underline{k}_\lambda[d_\lambda]$) where $j_\lambda : \mathcal{G}r^\lambda \to \mathcal{G}r$ denotes the inclusion, $\underline{k}_\lambda$ the constant sheaf on $\mathcal{G}r^\lambda$ and d_λ the dimension of $\mathcal{G}r^\lambda$.

We say $\mathcal{F} \in P(\mathcal{G}r)$ is a tilting sheaf if it corresponds to a tilting module for $\mathbf{G}$. This is equivalent to admitting two filtrations — one with standard successive quotients and the other with costandard successive quotients.[(4)] We denote by $\mathcal{T}(\lambda)$ the tilting sheaf corresponding to the indecomposable tilting module $T(\lambda)$.

Our main theorem is the following geometric characterization of the tilting sheaves on the affine Grassmannian. We will need to assume that the characteristic p is bigger than some bound depending only on the root system Φ of $\mathbf{G}$.

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(2) Unless stated otherwise, in this paper parity complexes are defined with respect to the constant pariversity $\natural$.

(3) This restriction can be removed by working in the non-equivariant setting.

(4) Warning: this definition of tilting sheaf is more general than that of [2], which does not apply to this setting.