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GEOMETRIC BAUM-CONNES ASSEMBLY MAP FOR TWISTED DIFFERENTIABLE STACKS

BY Paulo CARRILLO ROUSE AND Bai-Ling WANG

Abstract. – We construct the geometric Baum-Connes assembly map for twisted Lie groupoids, that means for Lie groupoids together with a given groupoid equivariant $PU(H)$-principle bundle. The construction is based on the use of geometric deformation groupoids, these objects allow in particular to give a geometric construction of the associated pushforward maps and to establish the functoriality. The main results in this paper are to define the geometric twisted K-homology groups and to construct the assembly map. Even in the untwisted case the fact that the geometric K-homology groups and the geometric assembly map are well defined for Lie groupoids is new, as it was only sketched by Connes in his book for general Lie groupoids without any restrictive hypothesis, in particular for non Hausdorff Lie groupoids.

We also prove the Morita invariance of the assembly map, giving thus a precise meaning to the geometric assembly map for twisted differentiable stacks. We discuss the relation of the assembly map with the associated assembly map of the S^1-central extension. The relation with the analytic assembly map is treated, as well as some cases in which we have an isomorphism. One important tool is the twisted Thom isomorphism in the groupoid equivariant case which we establish in the appendix.

Résumé. – Nous construisons le morphisme d'assemblage géométrique de Baum-Connes pour des groupoïdes de Lie tordus, à savoir des groupoïdes de Lie avec un $PU(H)$-fibré principal équivariant. La construction est basée sur l'utilisation des groupoïdes de déformation, ces objets permettent en particulier de donner une construction géométrique des morphismes shriek associés et d'établir la fonctorialité. Les principaux résultats de cet article sont la définition des groupes de K-homologie géométrique tordue et la construction du morphisme d'assemblage. Même dans le cas non tordu le fait que les groupes de K-homologie géométrique et le morphisme d'assemblage (géométrique) pour des groupoïdes de Lie sont bien définis est nouveau ; en effet, ceci a été esquissé par Connes dans son livre pour des groupoïdes de Lie générales sans aucune restriction, en particulier pour des groupoïdes non séparés.

Nous montrons aussi l'invariance par Morita du morphisme d'assemblage, donnant ainsi un sens précis au morphisme d'assemblage géométrique de Baum-Connes pour des champs différentiables tordus. Nous discutons la relation de notre morphisme d'assemblage avec le morphisme associé à la S^1-extension centrale. La relation avec le morphisme analytique est traitée, ainsi que quelques cas où il y a isomorphisme. Un outil important est le morphisme de Thom tordu dans le cas équivariant par rapport à un groupoïde que nous établissons dans l'appendice.

1. Introduction

The present paper is a natural sequel of [11] where we started a study of an index theory for foliations with the presence of $PU(H)$-twistings (see also [10]).

In [4] Baum and Connes introduced a geometrically defined K-theory for Lie groups, group actions and foliations. Its main features are its computability and simplicity of its definition, besides, in some cases they were able to construct a (also geometric) Chern character. Using classic ideas from index theory they constructed a natural map from this group to the analytic K-theory. This so-called Baum-Connes assembly map gave rise to many research developments due to its connection to many areas of mathematics and mathematical physics. Very interesting geometric and analytic corollaries can be deduced from the injectivity, surjectivity or bijectivity of the Baum-Connes map. Shortly after the paper by Baum-Connes, the powerful tools of KK-theory took over the originally geometrically defined map. Indeed, the use of KK-theory to define the assembly map have given extraordinary results. However the original geometrically defined map was somehow lost. In fact for some years experts assume both approaches to be the same but it took some years to give the actual proof for some cases.

The geometric approach is very interesting for several reasons, for instance the use of geometric K-homology in index theory and hence a completely geometric way of doing index theory, the possibility of defining a (geometric) Chern character from the geometric K-homology and hence to obtain explicit formulae. It is more suitable for geometric situation for which the analytic approach is not yet understood, for example for general Lie groupoids the analytic assembly is only defined for Hausdorff groupoids.

In this paper we construct the geometric Baum-Connes map for a twisted Lie groupoid $(\mathscr{G},\alpha)$, that is a Lie groupoid $\mathscr{G}$ together with a given equivariant (with respect to the groupoid action) $PU(H)-$principle bundle on $\mathscr{G}$. Equivalently, a twisting is given by a Hilsum-Skandalis morphism

$$\alpha : \mathscr{G} \dashrightarrow PU(H).$$

Even in the untwisted case this was not done before. In fact, in Connes book ([13] II.10.α), he proposes a definition for the geometric group of a Lie groupoid and he sketches the construction for the assembly map using deformation groupoids ideas that englobes what he did in [4] with Baum. We utilize these ideas to study the assembly map for the twisted case.

Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with a given twisting α on $\mathscr{G}$. Given such a data we can consider the maximal C^*-algebra $C^*(\mathscr{G},\alpha)$ (or reduced if indicated), the algebra is constructed by taking a S^1-central extension R_α associated to α via the canonical S^1-central extension $S^1 \longrightarrow U(H) \longrightarrow PU(H)$ and using one factor of the algebra associated to such extension(1); for complete details, see Section 3.1 below.

Now, consider a $\mathscr{G}$-manifold P with momentum map $\pi_P : P \longrightarrow M$ which is assumed to be a submersion. Denote by T^vP the vertical tangent bundle associated to π_P. In this paper we will assume that for any $\mathscr{G}$-manifold P, T^vP is an oriented vector bundle which admits a

(1) The extension depends of the choice of a cocycle defining α, however two such extensions are Morita equivalent via an explicit equivalence and hence the algebras they define are Morita equivalent as well.

$\mathscr{G}$-invariant metric, for instance when $\mathscr{G}$ acts on P properly or when $P = M$. We will denote the twisted analytic K-theory groups of the action groupoid $P \rtimes \mathscr{G}$ by

$$K^*(P \rtimes \mathscr{G}, \alpha) := K_{-*}(C^*(P \rtimes \mathscr{G}, \pi_P^*\alpha))$$

where $\pi_P^*\alpha$ is the pull-back twisting on $P \rtimes \mathscr{G}$ by the groupoid morphism $\pi_P : P \rtimes \mathscr{G} \longrightarrow \mathscr{G}$ (we use the same notation for π_P at the level of the arrows). One can consider a S^1-central extension R_α over a Cech groupoid $\mathscr{G}_\Omega$ (Morita equivalent to $\mathscr{G}$). If there is an extra twisting we will add it in the notation and explain it case by case.

Let P, N be two $\mathscr{G}$-manifolds and $f : P \longrightarrow N$ a $\mathscr{G}$-equivariant oriented smooth map. Using only geometric deformation groupoids, we construct a morphism[(2)] , the shriek map,

$$K^*(P \rtimes \mathscr{G}, \alpha + \mathfrak{o}_f) \xrightarrow{f_!} K^*(N \rtimes \mathscr{G}, \alpha) \tag{1.1}$$

where $\mathfrak{o}_f$ is the orientation twisting[(3)] over $P \rtimes \mathscr{G}$ of the $\mathscr{G}$-vector bundle $f^*T^vN \oplus T^vP$.

We remark that the construction of the shriek map is by means of deformation groupoids, this gives an explicit geometric pushforward map that gives exactly the corresponding equivariant family index when f is a submersion. Moreover, we establish the functoriality of the construction by again only using deformation groupoids, this gives a very geometric flavor to the proof, indeed one can understand the functoriality via a double deformation from one groupoid to another one. As we mentioned above, this was not done before even in the untwisted case, in fact, in [13] (Section II.6) Connes sketched the construction for the classic pushforward between manifolds using deformation groupoids and left the proof of the functoriality as an exercise. We remark that the result below (Theorem 4.2) was proved (for f, g submersions) using analytic methods by Tu and Xu ([41] 4.19), the statement is the following:

Theorem 1.1. – The push-forward morphism (1.1) is functorial, that means, if we have a composition of smooth $\mathscr{G}$-oriented smooth maps between $\mathscr{G}$-manifolds $P \xrightarrow{f} N \xrightarrow{g} L$, and a twisting $\alpha : \mathscr{G} - - \to PU(H)$, then the following diagram commutes

$$\begin{tikzcd} K^*(P \rtimes \mathscr{G}, \alpha + \mathfrak{o}_{g\circ f}) \arrow[rr, "(g\circ f)_!"] \arrow[dr, "f_!"'] & & K^*(L \rtimes \mathscr{G}, \alpha) \\ & K^*(N \rtimes \mathscr{G}, \alpha + \mathfrak{o}_g). \arrow[ur, "g_!"'] & \end{tikzcd}$$

The above theorem enables us to define the associated geometric K-homology group for a Lie groupoid with a twisting.

Definition 1.2 (Twisted geometric K-homology). – Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with a twisting $\alpha : \mathscr{G} - - - > PU(H)$. The twisted geometric K-homology group associated to $(\mathscr{G}, \alpha)$ is the abelian group denoted by $K_*^{\mathrm{geo}}(\mathscr{G}, \alpha)$ with generators and relations described as follows. A generator is called a cycle (P, ξ) where

(1) P is a smooth co-compact $\mathscr{G}$-proper manifold,

(2) In [11] the special case where $\mathscr{G}$ is the holonomy groupoid of a foliation and the action on P is free is treated, we proved there in particular the functoriality as an application of a longitudinal index theorem.
(3) in the sense of Example 2 in 2.12.