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ON THE CYCLE CLASS MAP FOR ZERO-CYCLES OVER LOCAL FIELDS

BY Hélène ESNAULT AND Olivier WITTENBERG
WITH AN APPENDIX BY Spencer BLOCH

Abstract. – We study the Chow group of 0-cycles of smooth projective varieties over local and strictly local fields. We prove in particular the injectivity of the cycle class map to integral ℓ-adic cohomology for a large class of surfaces with positive geometric genus, over local fields of residue characteristic $\neq \ell$. The same statement holds for semistable $K3$ surfaces defined over $\mathbf{C}((t))$, but does not hold in general for surfaces over strictly local fields.

Résumé. – Nous étudions le groupe de Chow des 0-cycles des variétés projectives et lisses sur les corps locaux et strictement locaux. Nous prouvons en particulier l'injectivité de l'application classe de cycle vers la cohomologie ℓ-adique entière pour de nombreuses surfaces de genre géométrique non nul, sur les corps locaux de caractéristique résiduelle $\neq \ell$. Le même énoncé vaut pour les surfaces $K3$ semi-stables définies sur $\mathbf{C}((t))$, mais ne vaut pas en général pour les surfaces sur les corps strictement locaux.

1. Introduction

Let X be a smooth projective variety over a field K, let $\mathrm{CH}_0(X)$ denote the Chow group of 0-cycles on X up to rational equivalence and let $A_0(X) \subset \mathrm{CH}_0(X)$ be the subgroup of cycle classes of degree 0.

When K is algebraically closed, the group $A_0(X)$ is divisible and its structure as an abelian group is, conjecturally, rather well understood, thanks to Roitman's theorem and to the Bloch-Beilinson-Murre conjectures. A central tool for the study of $A_0(X)$ over other types of fields is the cycle class map

$$\psi : \mathrm{CH}_0(X)/n\mathrm{CH}_0(X) \to H^{2d}_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}(d))$$

to étale cohomology, where n denotes an integer invertible in K and $d = \dim(X)$. The group $H^{2d}_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}(d))$ is easier to understand, thanks to the Hochschild-Serre spectral

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sequence; for instance, if K has cohomological dimension ≤ 1 and X is simply connected, then $H^{2d}_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z}(d)) = \mathbf{Z}/n\mathbf{Z}$ and ψ may be interpreted as the degree map.

According to one of the main results of higher-dimensional unramified class field theory, due to Kato and Saito [43], if K is a finite field, the group $A_0(X)$ is finite and ψ is an isomorphism. More recently, Saito and Sato [69] have shown that if K is the quotient field of an excellent Henselian discrete valuation ring with finite or separably closed residue field, the group $A_0(X)$ is the direct sum of a finite group of order prime to p and a group divisible by all integers prime to p (see also [11, Théorème 3.25]). In this case, however, the map ψ need not be either injective or surjective. What Saito and Sato prove, instead, is the bijectivity of the analogous cycle class map for cycles of dimension 1 on regular models of X over the ring of integers of K (see [69, Theorem 1.16]).

Following a method initiated by Bloch [4], one may approach the torsion subgroup of $A_0(X)$, as well as the kernel of ψ, with the help of algebraic K-theory, when X is a surface. We refer to [10] for a detailed account of this circle of ideas. Strong results were obtained in this way for rational surfaces, and more generally for surfaces with geometric genus zero, over number fields, p-adic fields, and fields of characteristic 0 and cohomological dimension 1 (see [5], [6], [16], [9], [14], [68]). We note that over algebraically closed fields of characteristic 0, surfaces with geometric genus zero are those surfaces for which the Chow group $A_0(X)$ should be representable, according to Bloch's conjecture (see [5, § 1]).

The first theorem of this paper establishes the injectivity of ψ for a large class of surfaces over local fields, when n is divisible enough and prime to the residue characteristic, without any assumption on the geometric genus. In principle, this theorem should be applicable to all simply connected surfaces, a generality in which the injectivity of ψ may not have been expected. Before we state it, we set up some notation. Let $\mathscr{X}$ be a regular proper flat scheme over an an excellent Henselian discrete valuation ring R. Let $X = \mathscr{X} \otimes_R K$ and $A = \mathscr{X} \otimes_R k$ denote the generic fiber and the special fiber, respectively. We assume the reduced special fiber A_{red} has simple normal crossings, and write $\mathrm{CH}_0(X) \mathbin{\widehat{\otimes}} \mathbf{Z}_\ell = \varprojlim \mathrm{CH}_0(X)/\ell^n \mathrm{CH}_0(X)$.

THEOREM 1 (Theorem 3.1 and Remark 3.2). – *Assume the residue field k is finite and X is a surface whose Albanese variety has potentially good reduction. If the irreducible components of A satisfy the Tate conjecture, then for any ℓ invertible in k, the cycle class map*

$$\mathrm{CH}_0(X) \mathbin{\widehat{\otimes}} \mathbf{Z}_\ell \to H^4_{\acute{e}t}(X, \mathbf{Z}_\ell(2)) \tag{1.1}$$

is injective. Equivalently, the natural pairing $\mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}$ is non-degenerate on the left modulo the maximal ℓ-divisible subgroup of $\mathrm{CH}_0(X)$.

The assumption on the irreducible components of A holds as soon as X has geometric genus zero, as well as in many examples of nontrivial degenerations of surfaces with nonzero geometric genus (see § 3). Theorem 1 is due to Saito [68] when X is a surface with geometric genus zero over a p-adic field. An example of Parimala and Suresh [62] shows that the assumption on the Albanese variety cannot be removed. Finally, we note that Theorem 1 may be viewed as a higher-dimensional generalization of Lichtenbaum-Tate duality for curves, according to which the natural pairing $\mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}$ is non-degenerate if X is a smooth proper curve over a p-adic field (see [54]).

Our starting point for the proof of Theorem 1 is the theorem of Saito and Sato alluded to above about the cycle class map for 1-cycles on $\mathscr{X}$ [69, Theorem 1.16], which allows us to express the kernel of ψ purely in terms of the scheme A and of the cohomology of X, when k is either finite or separably closed (Theorem 2.1 and Theorem 2.2). The dimension of X plays no role in this part of the argument; an application to the study of 0-cycles on a cubic threefold over $\mathbf{Q}_p$ may be found in Example 2.12. Theorem 1 is then obtained by analyzing the various cohomology groups which appear in the resulting expression for $\mathrm{Ker}(\psi)$. More precisely, in the situation of Theorem 1, we prove the stronger assertion that the 1-dimensional cycle class map $\psi_{1,A} : \mathrm{CH}_1(A) \mathbin{\widehat{\otimes}} \mathbf{Z}_\ell \to H^4_A(\mathscr{X}, \mathbf{Z}_\ell(2))$ to integral ℓ-adic étale homology is surjective. This provides a geometric explanation for the assumption that the Albanese variety of X have potentially good reduction, a condition which first appeared in [68] and which turns out to be essential for the surjectivity of $\psi_{1,A}$ to hold (see Lemma 3.7 and § 4.3).

When the residue field k is separably closed instead of finite, the arguments used in the proof of Theorem 1 fail in several places. They still lead to the following statement, which may also be deduced from results of Colliot-Thélène and Raskind [13] (see § 4 for comments on this point).

THEOREM 2 (Theorem 4.1 and Remark 4.3). – *Assume k is separably closed and K has characteristic* 0*. If X is a surface with geometric genus zero, then for any ℓ invertible in k, the cycle class map*

$$\mathrm{CH}_0(X) \mathbin{\widehat{\otimes}} \mathbf{Z}_\ell \to H^4_{\acute{e}t}(X, \mathbf{Z}_\ell(2)) \tag{1.2}$$

is injective. If in addition X is simply connected, then $A_0(X)$ is divisible by ℓ and the unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ vanishes.

This leaves open the question of the injectivity of the cycle class map (1.2) when k is separably closed and X is a surface with positive geometric genus over K. In this situation, the 1-dimensional cycle class map $\psi_{1,A}$ is far from being surjective. Building on the work of Kulikov, Persson, Pinkham [51] [64], and of Miranda and Morrison [59], we nevertheless give a positive answer for semistable $K3$ surfaces over $\mathbf{C}((t))$.

THEOREM 3 (Theorem 5.1). – *Let X be a $K3$ surface over $\mathbf{C}((t))$. If X has semistable reduction, the group $A_0(X)$ is divisible.*

The proof of Theorem 3 hinges on the precise knowledge of the combinatorial structure of a degeneration of X. It would go through over the maximal unramified extension of a p-adic field, as far as prime-to-p divisibility is concerned, if similar knowledge were available. This is in marked contrast with the situation over p-adic fields, where such knowledge is not necessary for the proof of Theorem 1.

In the final section of this paper, with the help of Ogg-Shafarevich theory and of a construction due to Persson, we show that the hope for a statement analogous to Theorem 1 over the quotient field of a strictly Henselian excellent discrete valuation ring is in fact too optimistic, even over the maximal unramified extension of a p-adic field.