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THE UNIVERSAL ABELIAN VARIETY OVER $\mathcal{A}_5$

BY GAVRIL FARKAS AND ALESSANDRO VERRA

ABSTRACT. – We establish a structure result for the universal abelian variety over $\mathcal{A}_5$. This implies that the boundary divisor of $\overline{\mathcal{A}}_6$ is unirational and leads to a lower bound on the slope of the cone of effective divisors on $\overline{\mathcal{A}}_6$.

RÉSUMÉ. – On établit un théorème de structure pour la variété abélienne universelle sur $\mathcal{A}_5$. Le résultat entraîne que le diviseur de la frontière de $\overline{\mathcal{A}}_6$ est unirationnel et ceci donne lieu à une borne inférieure pour la pente du cône des diviseurs effectifs en $\overline{\mathcal{A}}_6$.

The general principally polarized abelian variety $[A, \Theta] \in \mathcal{A}_g$ of dimension $g \leq 5$ can be realized as a Prym variety. Abelian varieties of small dimension can be studied in this way via the rich and concrete theory of curves, in particular, one can establish that $\mathcal{A}_g$ is unirational in this range. In the case $g = 5$, the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is finite of degree 27, see [7]; three different proofs [6, 17], [22] of the unirationality of $\mathcal{R}_6$ are known. The moduli space $\mathcal{A}_g$ is of general type for $g \geq 7$, see [12, 18], [21]. Determining the Kodaira dimension of $\mathcal{A}_6$ is a notorious open problem.

The aim of this paper is to give a simple unirational parametrization of the universal abelian variety over $\mathcal{A}_5$ and hence of the boundary divisor of a compactification of $\mathcal{A}_6$. We denote by $\phi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ the universal abelian variety of dimension $g - 1$ (in the sense of stacks). The moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g and their rank 1 degenerations is a partial compactification of $\mathcal{A}_g$ obtained by blowing up $\mathcal{A}_{g-1}$ in the Satake compactification, cf. [18]. Its boundary $\partial\widetilde{\mathcal{A}}_g$ is isomorphic to the universal Kummer variety in dimension $g - 1$ and there exists a surjective double covering $j : \mathcal{X}_{g-1} \to \partial\widetilde{\mathcal{A}}_g$. We establish a simple structure result for the boundary $\partial\widetilde{\mathcal{A}}_6$:

THEOREM 0.1. – *The universal abelian variety $\mathcal{X}_5$ is unirational.*

This immediately implies that the boundary divisor $\partial\widetilde{\mathcal{A}}_6$ is unirational as well. What we prove is actually stronger than Theorem 0.1. Over the moduli space $\mathcal{R}_g$ of smooth Prym curves of genus g, we consider the universal Prym variety $\varphi : \mathcal{Y}_g \to \mathcal{R}_g$ obtained by pulling back $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ via the Prym map $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Let $\overline{\mathcal{R}}_g$ be the moduli space of stable

Prym curves of genus g together with the universal Prym curve $\tilde{\pi}:\widetilde{\mathscr{C}}\to\overline{\mathscr{R}}_g$ of genus $2g-1$. If $\widetilde{\mathscr{C}}^{g-1}:=\widetilde{\mathscr{C}}\times_{\overline{\mathscr{R}}_g}\cdots\times_{\overline{\mathscr{R}}_g}\widetilde{\mathscr{C}}$ is the $(g-1)$-fold product, one has a universal *Abel-Prym* rational map $\mathfrak{ap}:\widetilde{\mathscr{C}}^{g-1}\dashrightarrow\mathscr{Y}_g$, whose restriction on each individual Prym variety is the usual Abel-Prym map. The rational map $\mathfrak{ap}$ is dominant and generically finite (see Section 4 for details). We prove the following result:

THEOREM 0.2. – *The five-fold product $\widetilde{\mathscr{C}}^5$ of the universal Prym curve over $\overline{\mathscr{R}}_6$ is unirational.*

The key idea in the proof of Theorem 0.2 is to view smooth Prym curves of genus 6 as discriminants of conic bundles, via their representation as symmetric determinants of quadratic forms in three variables. We fix four general points $u_1,\ldots,u_4\in\mathbf{P}^2$ and set $w_i:=(u_i,u_i)\in\mathbf{P}^2\times\mathbf{P}^2$. Since the action of the automorphism group $\mathrm{Aut}(\mathbf{P}^2\times\mathbf{P}^2)$ on $\mathbf{P}^2\times\mathbf{P}^2$ is 4-transitive, any set of four general points in $\mathbf{P}^2\times\mathbf{P}^2$ can be brought to this form. We then consider the linear system

$$\mathbf{P}^{15}:=\Big|\mathscr{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\Big|\subset\Big|\mathscr{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,2)\Big|$$

of hypersurfaces $Q\subset\mathbf{P}^2\times\mathbf{P}^2$ of bidegree $(2,2)$ which are nodal at $w_1,\ldots,w_4$. For a general threefold $Q\in\mathbf{P}^{15}$, the first projection $p:Q\to\mathbf{P}^2$ induces a conic bundle structure with a sextic discriminant curve $\Gamma\subset\mathbf{P}^2$ such that $p(\mathrm{Sing}(Q))=\mathrm{Sing}(\Gamma)$. The discriminant curve Γ is nodal precisely at the points $u_1,\ldots,u_4$. Furthermore, Γ is equipped with an unramified double cover $p_\Gamma:\widetilde{\Gamma}\to\Gamma$, parametrizing the lines which are components of the singular fibres of $p:Q\to\mathbf{P}^2$. By normalizing, p_Γ lifts to an unramified double cover $f:\widetilde{C}\to C$ between the normalization $\widetilde{C}$ of $\widetilde{\Gamma}$ and the normalization C of Γ respectively. Note that there exists an exact sequence of generalized Prym varieties

$$0\longrightarrow(\mathbf{C}^*)^4\longrightarrow P(\widetilde{\Gamma}/\Gamma)\longrightarrow P(\widetilde{C}/C)\longrightarrow 0.$$

One can show without much effort that the assignment $\mathbf{P}^{15}\ni Q\mapsto[\widetilde{C}\xrightarrow{f}C]\in\mathscr{R}_6$ is dominant. This offers an alternative, much simpler, proof of the unirationality of $\mathscr{R}_6$. However, much more can be obtained with this construction.

Let $\mathbf{G}:=\mathbf{P}^2\times(\mathbf{P}^2)^\vee=\big\{(o,\ell):o\in\mathbf{P}^2,\ell\in\{o\}\times(\mathbf{P}^2)^\vee\big\}$ be the Hilbert scheme of lines in the fibres of the first projection $p:\mathbf{P}^2\times\mathbf{P}^2\to\mathbf{P}^2$. Since containing a given line in a fibre of p imposes *three* linear conditions on the linear system $\mathbf{P}^{15}$ of threefolds $Q\subset\mathbf{P}^2\times\mathbf{P}^2$ as above, it follows that imposing the condition $\{o_i\}\times\ell_i\subset Q$ for *five* general lines, singles out a *unique* conic bundle $Q\in\mathbf{P}^{15}$. This induces an étale double cover $f:\widetilde{C}\to C$, as above, over a smooth curve of genus 6. Moreover, f comes equipped with five marked points $\ell_1,\ldots,\ell_5\in\widetilde{C}$. To summarize, we can define a rational map

$$\zeta:\mathbf{G}^5\dashrightarrow\widetilde{\mathscr{C}}^5,\quad\zeta\Big((o_1,\ell_1),\ldots,(o_5,\ell_5)\Big):=\Big(f:\widetilde{C}\to C,\ell_1,\ldots,\ell_5\Big),$$

between two 20-dimensional varieties, where $\mathbf{G}^5$ denotes the 5-fold product of $\mathbf{G}$.

THEOREM 0.3. – *The morphism $\zeta:\mathbf{G}^5\dashrightarrow\widetilde{\mathscr{C}}^5$ is dominant, so that $\widetilde{\mathscr{C}}^5$ is unirational.*

More precisely, we show that $\mathbf{G}^5$ is birationally isomorphic to the fibre product $\mathbf{P}^{15}\times_{\overline{\mathcal{R}}_6}\widetilde{\mathcal{C}}^5$. In order to set Theorem 0.3 on the right footing and in view of enumerative calculations, we introduce a $\mathbf{P}^2$-bundle $\pi : \mathbf{P}(\mathcal{M}) \to S$ over the quintic del Pezzo surface S obtained by blowing up $\mathbf{P}^2$ at the points $u_1, \ldots, u_4$. The rank 3 vector bundle $\mathcal{M}$ on S is obtained by making an elementary transformation along the four exceptional divisors $E_1, \ldots, E_4$ over $u_1, \ldots, u_4$. The nodal threefolds $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ considered above can be thought of as sections of a tautological linear system on $\mathbf{P}(\mathcal{M})$ and, via the identification

$$\left|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right| = \left|\mathcal{O}_{\mathbf{P}(\mathcal{M})}(2)\right|,$$

we can view 4-nodal conic bundles in $\mathbf{P}^2 \times \mathbf{P}^2$ as *smooth* conic bundles over S. In this way the numerical characters of a pencil of such conic bundles can be computed (see Sections 2 and 3 for details).

Theorem 0.3 is then used to give a lower bound for the slope of the effective cone of $\overline{\mathcal{A}}_6$ (though we stop short of determining the Kodaira dimension of $\overline{\mathcal{A}}_6$). Recall that if E is an effective divisor on the perfect cone compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ with no component supported on the boundary $D_g := \overline{\mathcal{A}}_g - \mathcal{A}_g$ and $[E] = a\lambda_1 - b[D_g]$, where $\lambda_1 \in CH^1(\widetilde{\mathcal{A}}_g)$ is the Hodge class, then the slope of E is defined as $s(E) := \frac{a}{b} \geq 0$. The slope $s(\overline{\mathcal{A}}_g)$ of the effective cone of divisors of $\overline{\mathcal{A}}_g$ is the infimum of the slopes of all effective divisors on $\overline{\mathcal{A}}_g$. This important invariant governs to a large extent the birational geometry of $\overline{\mathcal{A}}_g$; for instance, since $K_{\overline{\mathcal{A}}_g} = (g+1)\lambda_1 - [D_g]$, the variety $\overline{\mathcal{A}}_g$ is of general type if $s(\overline{\mathcal{A}}_g) < g+1$, and uniruled when $s(\overline{\mathcal{A}}_g) > g+1$. It is shown in the appendix of [14] that the slope of the moduli space $\overline{\mathcal{A}}_g$ is independent of the choice of a toroidal compactification.

It is known that $s(\overline{\mathcal{A}}_4) = 8$ and that the Jacobian locus $\overline{\mathcal{M}}_4 \subset \overline{\mathcal{A}}_4$ achieves the minimal slope [19]; one of the results of [9] is the calculation $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, the only irreducible effective divisor on $\overline{\mathcal{A}}_5$ of minimal slope is the closure of the Andreotti-Mayer divisor N_0' consisting of 5-dimensional ppav's $[A, \Theta]$ for which the theta divisor Θ is singular at a point which is not 2-torsion. Concerning $\overline{\mathcal{A}}_6$, we establish the following estimate:

THEOREM 0.4. – *The following lower bound holds:* $s(\overline{\mathcal{A}}_6) \geq \frac{53}{10}$.

Note that this is the first concrete lower bound on the slope of $\overline{\mathcal{A}}_6$. The idea of proof of Theorem 0.4 is very simple. Since $\widetilde{\mathcal{C}}^5$ is unirational, we choose a suitable sweeping rational curve $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$, which we then push forward to $\overline{\mathcal{A}}_6$ as follows:

$$\begin{tikzcd} \mathbf{P}^1 \arrow[r, "i"] \arrow[rrrr, bend left, "h"] & \widetilde{\mathcal{C}}^5 \arrow[r, "\mathfrak{ap}"] & \widetilde{\mathcal{Y}}_6 \arrow[r] & \widetilde{\mathcal{X}}_5 \arrow[r, "j"] & D_6. \end{tikzcd}$$

Here $\widetilde{\mathcal{Y}}_6$ and $\widetilde{\mathcal{X}}_5$ are partial compactifications of $\mathcal{Y}_6$ and $\mathcal{X}_5$ respectively which are described in Section 4, whereas D_6 is the boundary divisor of $\overline{\mathcal{A}}_6$. The curve $h(\mathbf{P}^1)$ sweeps the boundary divisor of $\overline{\mathcal{A}}_6$ and intersects non-negatively any effective divisor on $\overline{\mathcal{A}}_6$ not containing D_6. In particular,

$$s(\overline{\mathcal{A}}_6) \geq \frac{h(\mathbf{P}^1)\cdot[D_6]}{h(\mathbf{P}^1)\cdot\lambda_1}.$$