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REDUCTION OF SYMPLECTIC HOMEOMORPHISMS

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AND Sobhan SEYFADDINI

Abstract. – In [9], we proved that symplectic homeomorphisms preserving a coisotropic submanifold C, preserve its characteristic foliation as well. As a consequence, such symplectic homeomorphisms descend to the reduction of the coisotropic C.

In this article we show that these reduced homeomorphisms continue to exhibit certain symplectic properties. In particular, in the specific setting where the symplectic manifold is a torus and the coisotropic is a standard subtorus, we prove that the reduced homeomorphism preserves spectral invariants and hence the spectral capacity.

To prove our main result, we use Lagrangian Floer theory to construct a new class of spectral invariants which satisfy a non-standard triangle inequality.

Résumé. – Nous avons démontré dans [9], qu'un homéomorphisme symplectique qui laisse invariante une sous-variété coïsotrope C, préserve également son feuilletage caractéristique. Il induit donc un homéomorphisme sur la réduction symplectique de C.

Dans cet article, nous démontrons que l'homéomorphisme ainsi obtenu exhibe certaines propriétés symplectiques. En particulier, dans le cas où la variété symplectique ambiante est un tore et la sous-variété coïsotrope est un sous-tore standard, nous démontrons que l'homéomorphisme réduit préserve les invariants spectraux et donc aussi la capacité spectrale.

Pour démontrer notre résultat principal, nous construisons, à l'aide de l'homologie de Floer lagrangienne, une nouvelle famille d'invariants spectraux qui satisfont un nouveau type d'inégalité triangulaire.

1. Introduction

1.1. Context and main result

The main objects under study in this paper are symplectic homeomorphisms. Given a symplectic manifold (M, ω), a homeomorphism $\phi : M \to M$ is called a symplectic homeomorphism if it is the C^0-limit of a sequence of symplectic diffeomorphisms. This definition is motivated by a celebrated theorem due to Gromov and Eliashberg which asserts that if a symplectic homeomorphism ϕ is smooth, then it is a symplectic diffeomorphism in the usual sense: $\phi^*\omega = \omega$.

Understanding the extent to which symplectic homeomorphisms behave like their smooth counterparts constitutes the central theme of C^0-symplectic geometry. A recent discovery of Buhovsky and Opshtein suggests that these homeomorphisms are capable of exhibiting far more flexibility than symplectic diffeomorphisms: In [5], they construct an example of a symplectic homeomorphism of the standard $\mathbb{C}^3$ whose restriction to the symplectic subspace $\mathbb{C} \times \{0\} \times \{0\}$ is the contraction $(z, 0, 0) \mapsto (\frac{1}{2}z, 0, 0)$. Such behavior is impossible for a symplectic diffeomorphism but of course very typical for a volume-preserving homeomorphism. On the other hand, it is well-known that symplectic homeomorphisms are surprisingly rigid in comparison to volume-preserving maps. The following example of rigidity is the starting point of this article: Recall that a coisotropic submanifold is a submanifold $C \subset M$ whose tangent space, at every point of C, contains its symplectic orthogonal: $TC^\omega \subset TC$. Moreover, the distribution TC^ω is integrable and the foliation it spans is called the characteristic foliation of C.

THEOREM 1 ([9]). – *Let C be a smooth coisotropic submanifold of a symplectic manifold (M, ω). Let ϕ denote a symplectic homeomorphism. If $C' = \phi(C)$ is smooth, then it is coisotropic. Furthermore, ϕ maps the characteristic foliation of C to that of C'.*

Prior to the discovery of the above theorem, the special cases of Lagrangian submanifolds and hypersurfaces have been treated, respectively, by Laudenbach-Sikorav [11] and Opshtein [17].

We are now in position to describe the problem we are interested in. Denote by $\mathcal{F}$ and $\mathcal{F}'$, respectively, the characteristic foliations of the coisotropics C and C' from the above theorem. The reduced spaces $\mathcal{R} = C/\mathcal{F}$ and $\mathcal{R}' = C'/\mathcal{F}'$ are defined as the quotients of the coisotropic submanifolds by their characteristic foliations. These spaces are, at least locally, smooth manifolds and they can be equipped with natural symplectic structures induced by ω. Since $\phi(\mathcal{F}) = \mathcal{F}'$, the homeomorphism ϕ induces a homeomorphism $\phi_R : \mathcal{R} \to \mathcal{R}'$ of the reduced spaces. It is a classical fact that when ϕ is smooth, and hence symplectic, the reduced map ϕ_R is a symplectic diffeomorphism as well. It is therefore natural to ask whether the homeomorphism ϕ_R remains symplectic, in any sense, when ϕ is not assumed to be smooth. This is the question we seek to answer in this article.

We begin by first supposing that the reduction ϕ_R is smooth. It turns out that this scenario can be resolved rather easily using a result of [9].

PROPOSITION 2. – *Let C be a coisotropic submanifold whose reduction $\mathcal{R}$ is a symplectic manifold*(1)*, and ϕ be a symplectic homeomorphism. Assume that $C' = \phi(C)$ is smooth and therefore is coisotropic and admits a reduction $\mathcal{R}'$. Denote by $\phi_R : \mathcal{R} \to \mathcal{R}'$ the map induced by ϕ. Then, if ϕ_R is smooth, it is symplectic.*

We would like to point out that a similar result, with a similar proof, has already appeared in [5] (See Proposition 6).

(1) This is always locally true.

Proof. – We will prove that for any smooth function f_R on $\mathscr{R}'$, the Hamiltonian flow generated by the function $f_R \circ \phi_R$ is $\phi_R^{-1}\phi^t_{f_R}\phi_R$, where $\phi^t_{f_R}$ is the Hamiltonian flow generated by f_R. It is not hard to conclude from this that ϕ_R is symplectic: For example, it can easily be checked that ϕ_R preserves the Poisson bracket, i.e., $\{h_R \circ \phi_R, g_R \circ \phi_R\} = \{h_R, g_R\} \circ \phi_R$ for any two smooth functions h_R, g_R on $\mathscr{R}'$.

Let $f_R \colon \mathscr{R}' \to \mathbb{R}$ be smooth. We denote by $g_R \colon \mathscr{R} \to \mathbb{R}$ the function defined by $g_R = f_R \circ \phi_R$. Let f and g be any smooth lifts to M of f_R and g_R, respectively.

First, notice that by definition the restrictions to C of $f \circ \phi$ and g coincide. Since g is constant on the characteristic leaves of C, its Hamiltonian flow ϕ^t_g preserves C. Thus $H = (f \circ \phi - g) \circ \phi^t_g$ vanishes on C for all t. By [9, Theorem 3], the flow of the continuous Hamiltonian [(2)] H follows the characteristic leaves of C. On the other hand we know that this flow is given by the formula $\phi^t_H = (\phi^t_g)^{-1}\phi^{-1}\phi^t_f\phi$. This isotopy descends to the reduction $\mathscr{R}$ where it induces the isotopy $(\phi^t_{g_R})^{-1}\phi_R^{-1}\phi^t_{f_R}\phi_R$. But since ϕ^t_H follows characteristics it must descend to the identity. Hence $(\phi^t_{g_R})^{-1}\phi_R^{-1}\phi^t_{f_R}\phi_R = \mathrm{Id}$ as claimed. □

When ϕ_R is not assumed to be smooth, the situation becomes far more complicated. The question of whether or not ϕ_R is a symplectic homeomorphism seems to be very difficult and, at least currently, completely out of reach. Given the difficulty of this question, one could instead ask if there exist symplectic invariants which are preserved by reduced homeomorphisms. In this spirit, and since symplectic homeomorphisms are capacity preserving, Opshtein formulated the following a priori easier problem:

QUESTION 3. – *Is the reduction ϕ_R of a symplectic homeomorphism ϕ preserving a coisotropic submanifold always a capacity preserving homeomorphism?*

Partial positive results have been obtained by Buhovsky and Opshtein [5]. They proved in particular that in the case where C is a hypersurface, the map ϕ_R is a "non-squeezing map" in the sense that for every open set U containing a symplectic ball of radius r, the image $\phi_R(U)$ cannot be embedded in a symplectic cylinder over a 2-disk of radius $R < r$. This does not resolve Opshtein's question, but since capacity preserving maps are non-squeezing it does provide positive evidence for it. In the case of general coisotropic submanifolds, they conjecture that the same holds and indicate as to how one might approach this conjecture.

In this article, we work in the specific setting where M is the torus $\mathbb{T}^{2(k_1+k_2)}$ equipped with its standard symplectic structure and $C = \mathbb{T}^{2k_1+k_2} \times \{0\}^{k_2}$. The reduction of C is $\mathbb{T}^{2k_1}$ with its usual symplectic structure. Our main theorem shows that, in this setting, the reduced homeomorphism ϕ_R preserves certain symplectic invariants referred to as spectral invariants. This answers Opshtein's question positively, as it follows immediately that the spectral capacity is preserved by ϕ_R.

More precisely, for a time-dependent Hamiltonian H, denote by $c_+(H)$ the spectral invariant, defined by Schwarz [20], associated to the fundamental class of M. Roughly speaking, $c_+(H)$ is the action value at which the fundamental class $[M]$ appears in the Floer homology of H; see Equation (12) in Section 2.3 for the precise definition. (We should caution the reader that our notations and conventions are different than those of [20]. For example, $c_+(H)$ in this article corresponds to $c(1; H)$ in [20] where 1 is the generator

(2) The continuous function H generates a continuous flow in the sense defined by Müller and Oh [16].