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BLOCK DECOMPOSITION OF THE CATEGORY OF ℓ-MODULAR SMOOTH REPRESENTATIONS OF $\mathrm{GL}_n(\mathrm{F})$ AND ITS INNER FORMS

BY VINCENT SÉCHERRE AND SHAUN STEVENS

ABSTRACT. – Let F be a nonarchimedean locally compact field of residue characteristic p, let D be a finite dimensional central division F-algebra and let R be an algebraically closed field of characteristic different from p. To any irreducible smooth representation of $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$ with coefficients in R, we can attach a uniquely determined inertial class of supercuspidal pairs of G. This provides us with a partition of the set of all isomorphism classes of irreducible representations of G. We write $\mathscr{R}(\mathrm{G})$ for the category of all smooth representations of G with coefficients in R. To any inertial class Ω of supercuspidal pairs of G, we can attach the subcategory $\mathscr{R}(\Omega)$ made of smooth representations all of whose irreducible subquotients are in the subset determined by this inertial class. We prove that the category $\mathscr{R}(\mathrm{G})$ decomposes into the product of the $\mathscr{R}(\Omega)$'s, where Ω ranges over all possible inertial class of supercuspidal pairs of G, and that each summand $\mathscr{R}(\Omega)$ is indecomposable.

RÉSUMÉ. – Soit F un corps commutatif localement compact non archimédien de caractéristique résiduelle p, soit D une F-algèbre à division centrale de dimension finie et soit R un corps algébriquement clos de caractéristique différente de p. A toute représentation lisse irréductible du groupe $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$ à coefficients dans R correspond une classe d'inertie de paires supercuspidales de G. Ceci définit une partition de l'ensemble des classes d'isomorphisme de représentations irréductibles de G. Notons $\mathscr{R}(\mathrm{G})$ la catégorie des représentations lisses de G à coefficients dans R et, pour toute classe d'inertie Ω de paires supercuspidales de G, notons $\mathscr{R}(\Omega)$ la sous-catégorie formée des représentations lisses dont tous les sous-quotients irréductibles appartiennent au sous-ensemble déterminé par cette classe d'inertie. Nous prouvons que $\mathscr{R}(\mathrm{G})$ est le produit des $\mathscr{R}(\Omega)$, où Ω décrit les classes d'inertie de paires supercuspidales de G, et que chaque facteur $\mathscr{R}(\Omega)$ est indécomposable.

Introduction

When considering a category of representations of some group or algebra, a natural step is to attempt to decompose the category into *blocks*; that is, into subcategories which are indecomposable summands. Thus any representation can be decomposed uniquely as a direct sum of pieces, one in each block; any morphism comes as a product of morphisms, one in

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each block; and this decomposition of the category is the finest decomposition for which these properties are satisfied. Then a full understanding of the category is equivalent to a full understanding of all of its blocks.

In the case of representations of a finite group G, over an algebraically closed field R, there is always a block decomposition. In the simplest case, when the characteristic of R is prime to the order of G, this is particularly straightforward: all representations are semisimple so each block consists of representations isomorphic to a direct sum of copies of a fixed irreducible representation. In the general case, there is a well-developed theory, beginning with the work of Brauer and Nesbitt, and understanding the block structure is a major endeavor.

Now suppose G is the group of rational points of a connected reductive algebraic group over a nonarchimedean locally compact field F, of residue characteristic p. When R has characteristic zero, a block decomposition of the category $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ of smooth R-representations of G was given by Bernstein [1], in terms of the classification of representations of G by their cuspidal support. Any irreducible representations π of G is a quotient of some (normalized) parabolically induced representation $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$, with ϱ a cuspidal irreducible representation of a Levi subgroup M of G; the pair (M,ϱ) is determined up to G-conjugacy by π and is called its *cuspidal support*. Then each such pair (M,ϱ) determines a block, whose objects are those representations of G all of whose subquotients have cuspidal support $(\mathrm{M},\varrho\chi)$, for some unramified character χ of M.

One important tool in proving this block decomposition is the equivalence of the following two properties of an irreducible R-representation π of G:

- π is not a quotient of any properly parabolically induced representation; equivalently, all proper Jacquet modules of π are zero (π is *cuspidal*);
- π is not a *sub*quotient of any properly parabolically induced representation $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$ with ϱ an irreducible representation (π is *supercuspidal*).

When R is an algebraically closed field of positive characteristic different from p (the *modular* case), these properties are no longer equivalent and the methods used in the characteristic zero case cannot be applied. Instead, one can attempt to define the *supercuspidal support* of a smooth irreducible R-representation π of G: it is a pair (M,ϱ) consisting of an irreducible supercuspidal representation ϱ of a Levi subgroup M of G such that π is a *subquotient* of $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$. However, for a general group G, it is not known whether the supercuspidal support of a representation is well-defined up to conjugacy; indeed, the analogous question for finite reductive groups of Lie type is also open.

In any case, one can define the notion of an *inertial supercuspidal class* $\Omega = [\mathrm{M},\varrho]_{\mathrm{G}}$: it is the set of pairs (M',ϱ'), consisting of a Levi subgroup M' of G and a supercuspidal representation ϱ' of M', which are G-conjugate to $(\mathrm{M},\varrho\chi)$, for some unramified character χ of M. Given such a class Ω, we denote by $\mathscr{R}_{\mathrm{R}}(\Omega)$ the full subcategory of $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ whose objects are those representations all of whose subquotients are isomorphic to a subquotient of $\boldsymbol{i}_{\mathrm{M}'}^{\mathrm{G}}\varrho'$, for some $(\mathrm{M}',\varrho')\in\Omega$.

The main purpose of this paper is then to prove the following result:

THEOREM. – *Let* G *be an inner form of* $\mathrm{GL}_n(\mathrm{F})$ *and let* R *be an algebraically closed field of characteristic different from p. Then there is a block decomposition*

$$\mathscr{R}_{\mathrm{R}}(\mathrm{G}) = \prod_{\Omega} \mathscr{R}_{\mathrm{R}}(\Omega),$$

where the product is taken over all inertial supercuspidal classes.

This theorem generalizes the Bernstein decomposition in the case that R has characteristic zero, and also a similar statement, for general R, stated by Vignéras [24] in the split case $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$; however, the authors were unable to follow all the steps in [24] so our proof is independent, even if some of the ideas come from there.

Our proof builds on work of Mínguez and the first author [16, 15], in which they give a classification of the irreducible R-representations of G, in terms of supercuspidal representations, and of the supercuspidal representations in terms of the theory of types. In particular, they prove that supercuspidal support is well-defined up to conjugacy, so that the irreducible objects in $\mathscr{R}_{\mathrm{R}}(\Omega)$ are precisely those with supercuspidal support in Ω.

One question we do not address here is the structure of the blocks $\mathscr{R}_{\mathrm{R}}(\Omega)$. Given the explicit results on supertypes here, it is not hard to construct a progenerator Π for $\mathscr{R}_{\mathrm{R}}(\Omega)$ as a compactly-induced representation; for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ this was done (independently) by Guiraud [11] (for level zero blocks) and Helm [12]. Then $\mathscr{R}_{\mathrm{R}}(\Omega)$ is equivalent to the category of $\mathrm{End}_{\mathrm{G}}(\Pi)$-modules. In the case that R has characteristic zero, the algebra $\mathrm{End}_{\mathrm{G}}(\Pi)$ was described as a tensor product of affine Hecke algebras of type A in [22] (or [7] in the split case); indeed, we use this description in our proof here. For R an algebraic closure $\overline{\mathbf{F}}_\ell$ of a finite field of characteristic $\ell \neq p$, and a block $\mathscr{R}_{\mathrm{R}}(\Omega)$ with $\Omega = [\mathrm{GL}_n(\mathrm{F}), \varrho]_{\mathrm{GL}_n(\mathrm{F})}$, Dat [9] has described this algebra; it is an algebra of Laurent polynomials in one variable over the R-group algebra of a cyclic ℓ-group. It would be interesting to obtain a description in the general case.

We now describe the proof of the theorem, which relies substantially on the theory of semisimple types developed in [22] (see [7] for the split case). Given an inner form G of $\mathrm{GL}_n(\mathrm{F})$, in [22] the authors constructed a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, consisting of a compact open subgroup $\mathbf{J}$ of G and an irreducible complex representation $\boldsymbol{\lambda}$ of $\mathbf{J}$. This family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, called semisimple types, satisfies the following condition: for every inertial cuspidal class Ω, there is a semisimple type $(\mathbf{J}, \boldsymbol{\lambda})$ such that the irreducible complex representations of G with cuspidal support in Ω are exactly those whose restriction to $\mathbf{J}$ contains $\boldsymbol{\lambda}$.

In [16], Mínguez and the first author extended this construction to the modular case: they constructed a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, consisting of a compact open subgroup $\mathbf{J}$ of G and an irreducible modular representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, called semisimple supertypes. However, they did not give the relation between these semisimple supertypes and inertial supercuspidal classes of G. In this paper, we prove:

- for each inertial supercuspidal class Ω, there is a semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$ such that the irreducible R-representations of G with supercuspidal support in Ω are precisely those which appear as subquotients of the compactly induced representation $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$;