ISSN 0012-9593

quatrième série - tome 49 fascicule 4 juillet-août 2016

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Christopher FREI & Marta PIEROPAN

O-minimality on twisted universal torsors and Manin's conjecture over number fields

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 49, 2016, p. 757 à 811

O-MINIMALITY ON TWISTED UNIVERSAL TORSORS AND MANIN'S CONJECTURE OVER NUMBER FIELDS

BY CHRISTOPHER FREI AND MARTA PIEROPAN

ABSTRACT. – Manin's conjecture predicts the distribution of rational points on Fano varieties. Using explicit parameterizations of rational points by integral points on universal torsors and lattice-point-counting techniques, it was proved for several specific varieties over $\mathbb{Q}$, in particular del Pezzo surfaces. We show how this method can be implemented over arbitrary number fields, by proving Manin's conjecture for a singular quartic del Pezzo surface of type $\mathbf{A}_3 + \mathbf{A}_1$. The parameterization step is treated in high generality with the help of twisted integral models of universal torsors. To make the counting step feasible over arbitrary number fields, we deviate from the usual approach over $\mathbb{Q}$ by placing higher emphasis on the geometry of numbers in the framework of o-minimal structures.

RÉSUMÉ. – La conjecture de Manin prédit la répartition des points rationnels sur les variétés de Fano. Elle a été vérifiée pour plusieurs variétés sur $\mathbb{Q}$, en particulier certaines surfaces de del Pezzo, en utilisant des paramétrisations explicites des points rationnels par des points entiers sur des torseurs universels et des techniques de comptage de points de réseaux. On montre comment on peut appliquer cette méthode sur les corps de nombres quelconques, en démontrant la conjecture de Manin pour une surface de del Pezzo singulière de degré quatre et de type $\mathbf{A}_3 + \mathbf{A}_1$. La paramétrisation est présentée d'un point de vue général qui utilise des modèles entiers tordus de torseurs universels. Pour rendre possible le comptage sur les corps de nombres, on dévie de la procédure usuelle sur $\mathbb{Q}$ en mettant l'accent sur la géométrie des nombres dans le cadre des structures o-minimales.

1. Introduction

Let K be a number field and S the anticanonically embedded del Pezzo surface of degree 4 and type $\mathbf{A}_3 + \mathbf{A}_1$ given in $\mathbb{P}^4_K$ by the equations

$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0. \tag{1.1}$$

Let U be the complement of the lines in S, and let H be the anticanonical height on $S(K)$ induced by the Weil height on $\mathbb{P}^4(K)$,

$$H(x_0 : \cdots : x_4) := \prod_{v \in \Omega_K} \max\{|x_0|_v, \ldots, |x_4|_v\},$$

where Ω_K is the set of places of K and the normalized absolute values $|\cdot|_v$ are given as follows: let w be the place of $\mathbb{Q}$ below v and K_v (resp. $\mathbb{Q}_w$) the completion of K at v (resp. of $\mathbb{Q}$ at w). Then $|\cdot|_v := \left|N_{K_v|\mathbb{Q}_w}(\,\cdot\,)\right|_w$, where $|\cdot|_w$ is the usual real or p-adic absolute value on $\mathbb{Q}_w$. We investigate the counting function

$$N_{U,H}(B) := |\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leq B\}|.$$

Generalized versions [6, 45] of Manin's conjecture [28] predict an asymptotic formula

$$N_{U,H}(B) = c_{S,H} B(\log B)^5(1 + o(1)), \quad \text{as } B \to \infty,$$

with a positive constant $c_{S,H}$, which has been conjecturally interpreted in [43, 6, 45]. Our first main result is a proof of Manin's conjecture for S:

THEOREM 1.1. – *Let K be a number field of degree d, let S be given in $\mathbb{P}^4_K$ by* (1.1)*, let U be the complement of the lines in S, and let $\epsilon > 0$. As $B \to \infty$,*

$$N_{U,H}(B) = c_{S,H} B(\log B)^5 + O(B(\log B)^{5-1/d+\epsilon}),$$

with an explicit $c_{S,H} > 0$. This formula agrees with Peyre's refined version of Manin's conjecture [45, Formule empirique 5.1]*. The implicit constant in the error term depends on K and ϵ.*

We describe the constant $c_{S,H}$ explicitly later in this section. The special cases of Theorem 1.1 where $K = \mathbb{Q}$ or K is imaginary quadratic were proved in [20, 23]. A version of Manin's conjecture over arbitrary global function fields was proved for our surface S in [10].

Manin's conjecture is known in some general cases. For complete intersections of large dimension compared to their degree, it follows from an application of the Hardy-Littlewood circle method (cf. [43, 40]). Moreover, it has been proved for certain classes of Fano varieties with additional structure coming from actions of algebraic groups, using Langlands' work on Eisenstein series [28] or harmonic analysis on adelic points (for example, for toric varieties [5] and equivariant compactifications of additive groups [15]).

Other known cases of Manin's conjecture concern specific varieties of low dimension. Del Pezzo surfaces over $\mathbb{Q}$ have received the most attention: some milestones here are the first special cases of Manin's conjecture for (singular or nonsingular) del Pezzo surfaces of degrees 5 [11], 4 [13], 3 [14], and 2 [2] that are not covered by [5] or [15]. The method behind these results and many further proofs of Manin's conjecture for specific varieties over $\mathbb{Q}$ is by now classical. It is usually referred to as the *universal torsor method*.

A major drawback of this method is that almost all of its successful applications are restricted to varieties over $\mathbb{Q}$. Recently, Derenthal and the first-named author started a project with the aim to generalize the universal torsor method to number fields beyond $\mathbb{Q}$. So far, they were able to adapt the basic framework to imaginary quadratic fields [22], and to apply it to some singular del Pezzo surfaces of degrees 4 [23] and 3 [24] over imaginary quadratic fields. To our best knowledge, the only published proofs of Manin's conjecture for varieties over arbitrary number fields that can be interpreted as applications of the universal torsor method concern projective spaces $\mathbb{P}^n_K$ [50] and a specific toric variety [29], which are also covered by [5].

Theorem 1.1 is a first step to overcome this restriction. It is based on the universal torsor approach and is the first proof of Manin's conjecture over arbitrary number fields for a variety that is not included in the general results mentioned above (see [26]). One should note that the surface S is an equivariant compactification of a semidirect product $\mathbb{G}_a \rtimes \mathbb{G}_m$, so recent techniques of Tanimoto and Tschinkel [52] using harmonic analysis could also apply. So far, this was worked out only over $\mathbb{Q}$.

1.1. The universal torsor method

Universal torsors were introduced and studied by Colliot-Thélène and Sansuc [16, 17] to investigate arithmetic properties such as the Hasse principle and weak approximation. Salberger [49] was the first to apply them to Manin's conjecture (see also [44]). After Salberger's pioneering work, the universal torsor method became a prevalent tool to prove special cases of Manin's conjecture over $\mathbb{Q}$.

A typical application of the universal torsor method to a specific del Pezzo surface S consists essentially of two parts:

(a) Parameterizing the rational points on an open subset U by integral points on a universal torsor over a minimal desingularization $\widetilde{S} \to S$, subject to certain coprimality conditions, and lifting the height function to these points.
(b) Counting these integral points of bounded height, essentially replacing sums by integrals and estimating the difference.

A framework covering these parts in some generality was developed over $\mathbb{Q}$ in [20] and generalized in [22] to imaginary quadratic fields.

1.2. Parameterization

The minimal desingularization $\widetilde{S}$ of S is a smooth projective variety over K. For such a variety $\widetilde{S}$ and a torsor Y over $\widetilde{S}$ under an algebraic K-group G, a classical result of Colliot-Thélène and Sansuc [17] shows that there is a partition

$$\widetilde{S}(K) = \bigsqcup_{[\sigma]\in H^1(K,G)} {}_\sigma\pi({}_\sigma Y(K)),$$

for twists ${}_\sigma\pi : {}_\sigma Y \to \widetilde{S}$ of Y. The finest partitions of this kind are achieved if Y is a universal torsor. For quantitative problems such as Manin's conjecture, it is desirable to obtain a parameterization of $\widetilde{S}(K)$ by points with integral coordinates, which allows us to apply lattice-point-counting techniques. Such a parameterization was obtained by Salberger [49] for proper, smooth, split toric varieties X over $\mathbb{Q}$ with globally generated anticanonical sheaf. In this case, the partition induced by a model $\pi : \mathcal{Y} \to \mathcal{X}$ of a universal torsor $Y \to X$ is trivial:

$$X(\mathbb{Q}) = \pi(\mathcal{Y}(\mathbb{Z})). \tag{1.2}$$

Here, the fibers of π are just the orbits under the action of $\mathbb{G}_m^r(\mathbb{Z}) \cong (\mathbb{Z}^\times)^r$, where r is the rank of the Picard group of X. Hence, we obtain a $(2^r : 1)$-parameterization of $X(\mathbb{Q})$ by integral points, which reduces Manin's conjecture to a lattice-point-counting problem. In almost all applications of the universal torsor method to special cases of Manin's conjecture over $\mathbb{Q}$, a parameterization of the form (1.2) is constructed by elementary methods that essentially consist of removing greatest common divisors between existing coordinates by