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SEPARABLE EXTENSIONS IN TENSOR-TRIANGULAR GEOMETRY AND GENERALIZED QUILLEN STRATIFICATION

BY PAUL BALMER

ABSTRACT. – We exhibit a link between the Going-Up Theorem in commutative algebra and Quillen Stratification in modular representation theory. To this effect, we study the continuous map induced on spectra by a separable extension of tensor-triangulated categories. We determine the image of this map and bound the cardinality of its fibers by the degree of the extension. We then prove a weak form of descent, "up-to-nilpotence," which allows us to generalize Quillen Stratification to equivariant derived categories.

RÉSUMÉ. – Nous montrons un lien entre le théorème du *going-up* en algèbre commutative et le théorème de stratification de Quillen en théorie des représentations modulaires. Dans ce but, nous étudions l'application continue induite sur les spectres par une extension séparable de catégories triangulées tensorielles. Nous en déterminons l'image et bornons le cardinal de ses fibres par le degré de l'extension. Nous prouvons alors une forme faible de descente, « à nilpotence près », qui nous permet de généraliser la stratification de Quillen à d'autres catégories dérivées équivariantes.

Introduction

There are rich and growing connections between commutative algebra and modular representation theory, notably via homological methods. Some of these connections can be built using tensor-triangulated categories. Recall that tensor-triangulated categories also appear in several other settings, like motivic theory, equivariant stable homotopy theory, or Kasparov's KK-theory of C*-algebras, for instance. This framework is the backdrop of *tensor-triangular geometry*, or *tt-geometry* for short, see [2].

In the present work, we use tt-geometry to connect two classical and well-known results, namely the Going-Up Theorem in commutative algebra and Quillen's Stratification Theorem in modular representation theory. Let us first remind the reader:

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Theorem (Going-Up). – *Let $R \subset A$ be an integral extension of commutative rings, let $\mathfrak{q}$ be a prime in A and $\mathfrak{p}'$ a prime in R containing $\mathfrak{q} \cap R$; then there exists a prime $\mathfrak{q}'$ in A containing $\mathfrak{q}$ such that $\mathfrak{q}' \cap R = \mathfrak{p}'$. Further,* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is surjective and a weak form of injectivity holds, known as "Incomparability": if $\mathfrak{q} \subseteq \mathfrak{q}'$ are two primes in A such that $\mathfrak{q} \cap R = \mathfrak{q}' \cap R$ then $\mathfrak{q} = \mathfrak{q}'$.*

Theorem (Quillen Stratification [24]). – *Let G be a finite group and $\Bbbk$ be a field of positive characteristic p dividing the order of G. Let $\mathcal{V}_G := \mathrm{Proj}(\mathrm{H}^\bullet(G, \Bbbk))$ be the* projective support variety *of G. Then there is a canonical homeomorphism*

$$\operatorname*{colim}_{H \in \mathrm{Or}(G,\,\mathrm{Abelem})} \mathcal{V}_H \xrightarrow{\sim} \mathcal{V}_G \tag{0.1}$$

where $\mathrm{Or}(G, \mathrm{Abelem})$ *is the full subcategory of the orbit category of G on the elementary abelian p-subgroups of G (see Definition 4.6 if necessary).*

It is not obvious to the naked eye why the above two results should be related. Let us observe the tip of the iceberg: The Going-Up Theorem forces the rings A and R to have the same Krull dimension; similarly Quillen Stratification forces the Krull dimension of $\mathcal{V}_G$ to be the maximum of the dimensions of the $\mathcal{V}_H$, i.e., the p-rank of G minus one – an important application of [24].

Here, we prove an analogue of Going-Up in tt-geometry and show that it specializes to Quillen Stratification when applied to modular representation theory. This result illustrates the connections between the two subjects and the depth of tt-geometry. To go beyond unification and connection, we also prove some new results, namely we extend Quillen Stratification to any tensor-triangulated category receiving the derived category of G.

1. Statement of results

To do tt-geometry, one needs a tt-category $\mathscr{K}$. (Here, "tt" is short for "tensor-triangular" or "tensor-triangulated," as appropriate.) In commutative algebra, we use $\mathscr{K} = \mathrm{D}^{\mathrm{perf}}(R)$, the homotopy category of bounded complexes of finitely generated projective R-modules. In modular representation theory, we use $\mathscr{K} = \mathrm{stab}(\Bbbk G)$, the stable category of finitely generated $\Bbbk G$-modules modulo the projective ones.

To recover spaces like the affine scheme $\mathrm{Spec}(R)$ in the first example and the support variety $\mathcal{V}_G$ in the second, we use the *spectrum* $\mathrm{Spc}(\mathscr{K})$ of a tt-category $\mathscr{K}$. This fundamental tool of tt-geometry was introduced in [1] as the universal topological space in which objects x of $\mathscr{K}$ admit reasonable supports $\mathrm{supp}(x) \subseteq \mathrm{Spc}(\mathscr{K})$; see Rem. 2.6. The spectrum can also be constructed by means of *prime ideals* $\mathscr{P}$ in $\mathscr{K}$. By [1], it recovers the Zariski spectrum $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R)$ and the support variety $\mathrm{Spc}(\mathrm{stab}(\Bbbk G)) \cong \mathcal{V}_G$. This unification is the key to relate affine schemes and support varieties via tt-geometry. See more in Remark 2.7.

To exhibit a first analogy between Going-Up and Quillen Stratification, let us observe that both situations involve not only *one* tt-category but actually *two* (or more). In commutative algebra, it is rather obvious: We have the ring homomorphism $R \to A$, hence an extension-of-scalars $\mathrm{D}^{\mathrm{perf}}(R) \to \mathrm{D}^{\mathrm{perf}}(A)$ which is a tt-functor. In modular representation theory, we also have tt-functors $\mathrm{Res}^G_H : \mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ given by restriction from the group G

to its subgroups $H \leq G$, for instance the elementary abelian ones. Another key step in our discussion is to understand those restriction functors $\mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ as a form of "extension-of-scalars," similar to the extension-of-scalars of commutative algebra.

This is made possible thanks to a good notion of "ring" in a general tt-category $\mathscr{K}$, inspired by commutative algebra but flexible enough to be useful in representation theory as well. These good rings are the "tt-rings" of [5]: An associative and unital ring object $A \otimes A \xrightarrow{\mu} A$ in $\mathscr{K}$ is called a *tt-ring* if it is commutative and *separable*, in the classical sense (i.e., the multiplication μ has a section $\sigma : A \to A \otimes A$ which is A-linear on both sides). Separability of A guarantees that the category $A\text{-Mod}_{\mathscr{K}}$ of good-old A-modules in $\mathscr{K}$ remains triangulated, by [3] (see Remark 2.14). Commutativity of A allows us to equip $A\text{-Mod}_{\mathscr{K}}$ with a tensor $\otimes_A$. In short, if A is a tt-ring in a tt-category $\mathscr{K}$, then $A\text{-Mod}_{\mathscr{K}}$ is again a tt-category and extension-of-scalars $F_A : \mathscr{K} \longrightarrow A\text{-Mod}_{\mathscr{K}}$ is a tt-functor.

By [6] the restriction functor $\mathrm{Res}^G_H : \mathrm{stab}(\Bbbk G) \to \mathrm{stab}(\Bbbk H)$ is actually such an extension-of-scalars with respect to the particular tt-ring $A^G_H := \Bbbk(G/H)$, with multiplication extending $\Bbbk$-linearly the rule $\gamma \cdot \gamma = \gamma$ and $\gamma \cdot \gamma' = 0$ for all $\gamma \neq \gamma'$ in G/H (see Constr. 4.1). More precisely, there is an equivalence

$$A^G_H\text{-Mod}_{\mathscr{K}(G)} \cong \mathscr{K}(H) \tag{1.1}$$

for $\mathscr{K}(G) = \mathrm{stab}(\Bbbk G)$ and of course $\mathscr{K}(H) = \mathrm{stab}(\Bbbk H)$; under this equivalence, extension-of-scalars $F_{A^G_H}$ becomes isomorphic to restriction Res^G_H. In particular, the induced map on spectra recovers the usual map on support varieties $\mathcal{V}_H \to \mathcal{V}_G$. This recasting of restriction as an extension-of-scalars is already true for derived categories, i.e., the above (1.1) holds if $\mathscr{K}(G)$ means $\mathrm{D^b}(\Bbbk G)$, etc. See details in [6, Part I]. Furthermore we show in [7] that such results hold quite generally for equivariant tt-categories, way beyond representation theory.

In general, for any tt-category $\mathscr{K}$ and any tt-ring A in $\mathscr{K}$, the tt-functor $F_A : \mathscr{K} \to A\text{-Mod}_{\mathscr{K}}$ induces a continuous map on spectra

$$\varphi_A := \mathrm{Spc}(F_A) \;:\; \mathrm{Spc}(A\text{-Mod}_{\mathscr{K}}) \longrightarrow \mathrm{Spc}(\mathscr{K})\,. \tag{1.2}$$

It is clearly important to study this map φ_A in general, independently of Going-Up or Quillen Stratification. Here is our main tt-geometric result (see details below):

1.3. Theorem (Descent-up-to-nilpotence, see Thm. 3.19). – *Let $\mathscr{K}$ be a tt-category and let A be a tt-ring in $\mathscr{K}$. Suppose that A has finite degree and that A detects $\otimes$-nilpotence of morphisms. Then we have a coequalizer of topological spaces*

$$\mathrm{Spc}(A^{\otimes 2}\text{-Mod}_{\mathscr{K}}) \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \mathrm{Spc}(A\text{-Mod}_{\mathscr{K}}) \xrightarrow{\varphi_A} \mathrm{Spc}(\mathscr{K}) \tag{1.4}$$

where φ_1 and φ_2 are induced by the two obvious homomorphisms $A \rightrightarrows A \otimes A$.

The *degree* of a tt-ring A has also been introduced in [5] but can easily be considered as a black-box here. It is the natural measure of the "size" of the tt-ring A. It will allow us to prove some results by induction on the degree, via Theorem 2.19. Having finite degree is a mild hypothesis which holds for all tt-rings in all standard tt-categories of compact objects in algebraic geometry, homotopy theory or modular representation theory, by [5, § 4]. See more in Section 2.