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ABUNDANCE FOR KÄHLER THREEFOLDS

BY Frédéric CAMPANA, Andreas HÖRING
AND Thomas PETERNELL

Abstract. – Let X be a compact Kähler threefold with terminal singularities such that K_X is nef. We prove that K_X is semiample, i.e., some multiple mK_X is generated by global sections.

Résumé. – Soit X une variété kählérienne compacte à singularités terminales. Si K_X est nef, nous montrons que K_X est semi-ample, c'est-à-dire qu'un multiple mK_X est engendré par ses sections globales.

1. Introduction

1.A. Main results

Since the 1990's, the minimal model program for smooth complex projective threefolds is complete: every such manifold X admits a birational model X', which is $\mathbb{Q}$-factorial with only terminal singularities such that either

- X' carries a Fano fibration, in particular, X' is uniruled, or
- the canonical bundle $K_{X'}$ is semi-ample, i.e., some positive multiple $mK_{X'}$ is generated by global sections.

There are basically two parts in the program: first to establish the existence of a model X' which is either a Mori fibre space or has nef canonical divisor, and then to show that nefness implies semi-ampleness. This second part, known as "abundance", is established by [53, 42, 49].

The aim of the present paper is to fully establish the minimal model program in the category of Kähler threefolds. The first part of the program, i.e., the existence of a bimeromorphic model X' which either is a Mori fibre space or has nef canonical divisor, was carried out in the papers [38] and [37]. Thus it remains to show that nefness of the canonical divisor implies semi-ampleness, i.e., abundance holds for Kähler threefolds:

THEOREM 1.1. – *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is nef. Then K_X is semi-ample, that is some positive multiple mK_X is globally generated.*

The paper [24] established the existence of some section in mK_X for non-algebraic minimal models, so the assumption above implies $\kappa(X) \geq 0$. In [60] abundance was shown for non-algebraic minimal models, excluding however the case when X has no non-constant meromorphic function. Our arguments do not use any information on the structure of X and work both for algebraic and non-algebraic Kähler threefolds.

As a corollary, we establish a longstanding conjecture [26] on Kähler threefolds:

THEOREM 1.2. – *Let X be a smooth compact Kähler threefold. Assume that X is simple, i.e., there is no positive-dimensional proper subvariety through a very general point of X. Then there exists a bimeromorphic morphism $X \to T/G$ where T is a torus and G a finite group acting on T.*

1.B. Outline of the paper

Let X be a normal compact $\mathbb{Q}$-factorial Kähler threefold X with only terminal singularities and nef canonical divisor K_X. Denote by $\kappa(X)$ its Kodaira dimension and by $\nu(X)$ the numerical dimension, which is defined as

$$\nu(X) := \max\{m \in \mathbb{N} \mid c_1(K_X)^m \not\equiv 0\}.$$

Both invariants are subject to the inequality $\kappa(X) \leq \nu(X)$, with equality if K_X is semi-ample. Conversely, as Kawamata observed in [39, Thm. 6.1], in order to prove abundance, it is sufficient to prove equality: $\kappa(X) = \nu(X)$. By the base-point free theorem and an argument of Kawamata [39, Thm. 7.3] the main challenge is to rule out the possibility

$$\kappa(X) = 0 \text{ and } 0 < \nu(X) < 3.$$

Since we know that $\kappa(X) \geq 0$, there exists a positive number m and an effective divisor D such that $D \in |mK_X|$. A natural way to prove that $\kappa(X) \geq 1$ is to consider the restriction map

$$r : H^0(X, d(m+1)K_X) \to H^0(D, dK_D).$$

Arguing by induction on the dimension we want to prove that $H^0(D, dK_D) \neq 0$ for some $d \in \mathbb{N}$ and that some non-zero section $u \in H^0(D, dK_D)$ lifts via r to a global section $\tilde{u} \in H^0(X, d(m+1)K_X)$ on X. Since D might be very singular it is however not possible to analyse the divisor D directly. In order to circumvent this difficulty, Kawamata [42] developed the strategy, further explored in [49], to consider log pairs (X, B) with $B = \operatorname{Supp} D$ and to improve the singularities of this pair via certain birational transformations. This requires deep techniques of birational geometry of pairs within the theory of minimal models. In particular we have to run a log MMP for certain log pairs (X, Δ).

Therefore the first part of the paper (Sections 3 and 4) establishes the foundations for a minimal model program for log pairs on Kähler threefolds. As a first step we prove the cone theorem for the dual Kähler cone:

THEOREM 1.3. – *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold that is not uniruled. Let Δ be an effective $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is dlt. Then there exists an at most countable family $(\Gamma_i)_{i \in I}$ of rational curves on X such that*

$$0 < -(K_X + \Delta) \cdot \Gamma_i \leq 4$$

and

$$\overline{NA}(X) = \overline{NA}(X)_{(K_X+\Delta)\geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

The dual Kähler cone $\overline{\mathrm{NA}}(X)$ replaces the Mori cone of curves $\overline{\mathrm{NE}}(X)$ which is obviously too small in the non-algebraic setting (cf. [38, Sect. 1]). Our result is actually more general: first we prove a weak form of the cone theorem for lc pairs (X, Δ) such that X has rational singularities (cf. Theorem 4.1). Then we derive Theorem 1.3 on page 992 as a consequence of the weak cone theorem and some contraction results. The second step is to prove that the $(K_X + \Delta)$-negative extremal rays appearing in the cone theorem can be contracted by a bimeromorphic morphism.

THEOREM 1.4. – *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold that is not uniruled. Let Δ be an effective $\mathbb{Q}$-divisor such that the pair (X, Δ) is dlt. Let R be a $(K_X + \Delta)$-negative extremal ray in $\overline{NA}(X)$. If the ray R is divisorial with exceptional divisor S (cf. Definition 4.2), suppose that S has slc singularities.*

Then the contraction of R exists in the Kähler category.

These two theorems generalize the analogous statements for terminal threefolds [38, Thm. 1.2,Thm. 1.3]. While the global strategy of the proofs is similar to [38], it is not an extension of earlier work: in this paper we have to deal with threefolds with non-isolated singularities. This leads to new geometric problems (cf. the proof of Theorem 4.2) and is more representative of the challenges that appear in higher dimension.

Based on these results we show in Section 6 how to replace the threefold X with some bimeromorphic model having a pluricanonical divisor $D \in |mK_X|$ such that D_{red} is not too singular. In this section we follow the arguments of [49, Ch. 13, 14], and we also address some technical points which do not appear in the literature. While these reduction steps assure the existence of some effective pluricanonical divisor on D_{red}, it is not obvious that it extends to a pluricanonical divisor on X (cf. [22] for recent progress in the projective case). Following a deformation argument of Miyaoka [54] [49, Ch. 11], the case $\kappa(X) = 0, \nu(X) = 1$ can be excluded without too much effort.

For the case $\nu(X) = 2$ the idea is to prove via a Riemann-Roch computation that $h^0(X, mK_X)$ grows linearly. Establishing the results necessary for this Riemann-Roch computation is the second part of the paper (Sections 7 and 8). First we extend Enoki's theorem [25] to show that the cotangent sheaf of a non-uniruled threefold with canonical singularities is generically nef, then we want to use this positivity result to prove that $(K_X + B) \cdot \hat{c}_2(X) \geq 0$ where $K_X + B$ is a nef log-canonical divisor and $\hat{c}_2(X)$ is the second Chern class of the Q-sheaf $\hat{\Omega}_X$. Since X is not smooth in codimension two, the proof of this inequality in the projective case ([42], [49, Ch. 14]) uses some involved computation for the second Todd class of the Q-sheaf $\hat{\Omega}_X$. We give a new argument which should be of