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BROWN'S DIHEDRAL MODULI SPACE AND FREEDOM OF THE GRAVITY OPERAD

BY JOHAN ALM AND DAN PETERSEN

ABSTRACT. – Francis Brown introduced a partial compactification $M_{0,n}^{\delta}$ of the moduli space $M_{0,n}$. We prove that the gravity cooperad, given by the degree-shifted cohomologies of the spaces $M_{0,n}$, is cofree as a nonsymmetric anticyclic cooperad; moreover, the cogenerators are given by the cohomology groups of $M_{0,n}^{\delta}$. As part of the proof we construct an explicit diagrammatically defined basis of $H^{\bullet}(M_{0,n})$ which is compatible with cooperadic cocomposition, and such that a subset forms a basis of $H^{\bullet}(M_{0,n}^{\delta})$. We show that our results are equivalent to the claim that $H^k(M_{0,n}^{\delta})$ has a pure Hodge structure of weight $2k$ for all k, and we conclude our paper by giving an independent and completely different proof of this fact. The latter proof uses a new and explicit iterative construction of $M_{0,n}^{\delta}$ from $\mathbb{A}^{n-3}$ by blow-ups and removing divisors, analogous to Kapranov's and Keel's constructions of $\overline{M}_{0,n}$ from $\mathbb{P}^{n-3}$ and $(\mathbb{P}^1)^{n-3}$, respectively.

RÉSUMÉ. – Francis Brown a introduit une compactification partielle $M_{0,n}^{\delta}$ de l'espace de modules $M_{0,n}$. Nous démontrons que la coopérade gravité, définie par la cohomologie (décalée en degré) des espaces $M_{0,n}$, est colibre comme coopérade non symétrique anti-cyclique; de plus, les cogénérateurs sont donnés par les groupes de cohomologie de $M_{0,n}^{\delta}$. La preuve construit une base explicite de $H^{\bullet}(M_{0,n})$ en termes de diagrammes. Cette base est compatible avec la cocomposition coopéradique, et admet un sous-ensemble qui est une base de $H^{\bullet}(M_{0,n}^{\delta})$. Nous montrons que nos résultats sont équivalents au fait que $H^k(M_{0,n}^{\delta})$ a une structure de Hodge pure de poids $2k$ pour tout k, et nous donnons de plus dans notre article une seconde preuve, plus directe, de ce dernier fait. Cette seconde preuve utilise une construction itérative nouvelle et explicite de $M_{0,n}^{\delta}$ à partir de $\mathbb{A}^{n-3}$ par éclatements et enlèvements de diviseurs, qui est analogue aux constructions de Kapranov et Keel de $\overline{M}_{0,n}$, respectivement à partir de $\mathbb{P}^{n-3}$ et $(\mathbb{P}^1)^{n-3}$.

Introduction

Let $M_{0,n}$ for $n \geq 3$ be the moduli scheme of n distinct ordered points on $\mathbb{P}^1$ up to the action of PGL_2, and $\overline{M}_{0,n}$ its Deligne-Mumford compactification. These are smooth affine (resp. projective) varieties over $\mathbf{Q}$ (or $\mathbf{Z}$) of dimension $(n-3)$. Motivated by the study of multiple zeta values, Brown introduced an intermediate space $M_{0,n} \subset M_{0,n}^{\delta} \subset \overline{M}_{0,n}$, depending

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on a dihedral structure δ on the set $\{1,\ldots,n\}$; that is, an identification with the integers from 1 to n with the edges of some unoriented n-gon. The space $M^{\delta}_{0,n}$ is again affine, and the union of all spaces $M^{\delta}_{0,n}$ over all dihedral structures constitutes an open affine covering of the scheme $\overline{M}_{0,n}$. In more detail, let $X^{\delta}_n \subset \overline{M}_{0,n}(\mathbf{R})$ be the closure of the cell parametrizing n distinct points on the circle $\mathbb{P}^1(\mathbf{R})$, ordered compatibly with the chosen dihedral structure δ. Then $M^{\delta}_{0,n}$ is the subvariety of $\overline{M}_{0,n}$ formed by adding to $M_{0,n}$ only those boundary divisors that have nonempty intersection with X^{δ}_n.

The relevance of $M^{\delta}_{0,n}$ in the theory of periods and multiple zeta values resides on the following. By Grothendieck's theorem on algebraic de Rham cohomology, the cohomology of $M_{0,n}$ can be computed using the global sections of the complex of algebraic differential forms. It is thus interesting to study integrals of the form

$$\int_{X^{\delta}_n} \omega$$

where $[\omega]$ is any top degree cohomology class. Such integrals typically diverge, since the form ω may have poles along the boundary of X^{δ}_n; the integral converges precisely when $[\omega]$ is in the image of the restriction map $H^{n-3}(M^{\delta}_{0,n}) \to H^{n-3}(M_{0,n})$. Brown proved that any relative period integral of $M_{0,n}$ (in the sense of Goncharov and Manin) can be decomposed as a $\mathbf{Q}[2i\pi]$-linear combination of integrals of this form, with ω defined over $\mathbf{Q}$. Moreover, each such integral evaluates to a rational linear combination of multiple zeta values. The cohomology groups $H^k(M^{\delta}_{0,n})$ and their Hodge structures are thus relevant to our understanding of motives and periods.

The degree-shifted cohomologies $\{H^{\bullet-1}(M_{0,n})\}_{n\geq 3}$ constitute an (anti)cyclic cooperad with Poincaré residue as cocomposition. This cooperad was introduced by Getzler, who called it the *gravity cooperad*, and we denote it coGrav. The homologies $\{H_\bullet(\overline{M}_{0,n})\}_{n\geq 3}$ constitute a cyclic operad with composition given, simply, by the maps induced by inclusions of boundary strata of $\overline{M}_{0,n}$. This operad is known as the *hypercommutative operad*, Hycom, and features prominently in Gromov-Witten theory. Ginzburg, Kapranov and Getzler have shown that the two are interchanged by Koszul duality: in particular, there is a quasi-isomorphism $\Omega^{\mathrm{cyc}}\mathsf{coGrav} \to \mathsf{Hycom}$ between the cyclic cobar construction on the gravity cooperad and the hypercommutative operad. The statement is, in a sense, encoded by the geometry of $\overline{M}_{0,n}$. The set of complex points decomposes as a union

$$\overline{M}_{0,n}(\mathbf{C}) = \coprod_{T\in\mathrm{Tree}_n} \prod_{v\in\mathrm{Vert}(T)} M_{0,n(v)}(\mathbf{C})$$

of strata labeled by trees. This decomposition says that $\{\overline{M}_{0,n}(\mathbf{C})\}_{n\geq 3}$ is the free cyclic operad of sets generated by the collection $\{M_{0,n}(\mathbf{C})\}_{n\geq 3}$ of points of the open moduli spaces. Once we include topology and go from sets to varieties it is no longer a free operad; instead the decomposition is (morally speaking) transformed into said Koszul duality relation.

Brown's partial compactification has a similar structure:

$$M^{\delta}_{0,n}(\mathbf{C}) = \coprod_{T\in\mathrm{PTree}_n} \prod_{v\in\mathrm{Vert}(T)} M_{0,n(v)}(\mathbf{C})$$

is now a union over strata indexed by *planar* trees, which can be read as saying that $\{M^{\delta}_{0,n}(\mathbf{C})\}_{n\geq 3}$ is the *planar* operad of sets freely generated by the collection $\{M_{0,n}(\mathbf{C})\}_{n\geq 3}$.

What we here term a planar operad might also be called a nonsymmetric cyclic operad. We call them planar because they are encoded by the combinatorics of planar (non-rooted) trees, just like cyclic operads are encoded by trees, operads by rooted trees, and nonsymmetric operads by planar rooted trees.

Note that, by Poincaré duality we could equally well take the hypercommutative operad as defined by $\mathsf{Hycom}_n = H^{\bullet-2}(\overline{M}_{0,n})$, with Gysin maps as composition. Analogously, the collection $\mathsf{Prim}_n = H^{\bullet-2}(M^{\delta}_{0,n})$ is an operad. Our first statement says that coGrav and Prim satisfy a duality relation of planar (co)operads, analogous to the duality relation of cyclic (co)operads between coGrav and Hycom.

THEOREM 0.1. – *The planar cobar construction* $\Omega^{\mathrm{pl}}\mathsf{coGrav}$ *and* Prim *are quasi-isomorphic as planar operads if and only if the mixed Hodge structure on* $H^k(M^{\delta}_{0,n})$ *is pure of weight* $2k$. *Moreover, the compositions of* Prim *are all zero, so either condition is equivalent to the statement that* coGrav *is (noncanonically) isomorphic to the cofree cooperad cogenerated by* Prim *(with degree shifted by one).*

We remark that we throughout write "cofree cooperad" for what should properly be called "cofree conilpotent cooperad"; we assume all cooperads to be conilpotent.

In the second and third parts of the paper we give independent proofs of the two properties mentioned. This may be logically redundant (the properties are, after all, equivalent), but we believe the proofs to be of independent interest. The second part is devoted to proving the following:

THEOREM 0.2. – *The gravity operad is the linear hull of a free nonsymmetric operad of sets.*

Thus coGrav, the linear dual of the gravity operad, is cofree on the (dual of the) linear hull of the generators of said nonsymmetric operad of sets. That we have to switch to the gravity *operad* at this point (and this point only) is an unfortunate minor hiccup, but it is necessary: coGrav is conilpotent, so it could not possibly be the linear hull of any kind of cooperad of sets. On the other hand we want to compute with differential forms and residues throughout, and the arguments are naturally formulated in terms of the cohomology of $M_{0,n}$. Thus working with the gravity operad rather than coGrav throughout would have been somewhat cumbersome.

In any case, this implies that $\Omega^{\mathrm{pl}}\mathsf{coGrav}$ and Prim are quasi-isomorphic as planar operads, but can also be regarded as showing something stronger. In particular, the result involves construction of an explicit basis $\{\alpha_G\}$ of $H^\bullet(M_{0,n})$, with a subset $\{\alpha_P\} \subset \{\alpha_G\}$ forming a basis for the image of $H^\bullet(M^{\delta}_{0,n})$ in $H^\bullet(M_{0,n})$.

In the third and final part we give a direct proof of:

THEOREM 0.3. – *The mixed Hodge structure on* $H^k(M^{\delta}_{0,n})$ *is pure of weight* $2k$.

The proof relies on an inductive construction of $M^{\delta}_{0,n}$ from $\mathbb{A}^{n-3}$, alternating between blowing up a smooth subvariety and then removing the strict transform of a divisor containing the blow-up center. It is inspired by Hassett's work on moduli spaces of weighted pointed stable curves. This construction of $M^{\delta}_{0,n}$ is new.

Our results have several interesting consequences. Let us begin with a rather immediate one: