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A "STRANGE" FUNCTIONAL EQUATION FOR EISENSTEIN SERIES AND MIRACULOUS DUALITY ON THE MODULI STACK OF BUNDLES

BY DENNIS GAITSGORY

ABSTRACT. – We show that the failure of the usual Verdier duality on Bun_G leads to a new duality functor on the category of D-modules, and we study its relation to the operation of Eisenstein series.

RÉSUMÉ. – Dans cet article, on démontre que la dualité de Verdier habituelle ne tenant pas pour le champ Bun_G, on peut la remplacer par un autre foncteur de dualité. On étudie la relation entre celui-ci et le foncteur de série d'Eisenstein.

Introduction

0.1. Context for the present work

0.1.1. – This paper arose in the process of developing what V. Drinfeld calls *the geometric theory of automorphic functions*. I.e., we study *sheaves* on the moduli stack Bun_G of principal G-bundles on a curve X. Here and elsewhere in the paper, we fix an algebraically closed ground field k, and we let G be a reductive group and X a smooth and complete curve over k.

In the bulk of the paper we will take k to be of characteristic 0, and by a "sheaf" we will understand an object of the derived category of D-modules. However, with appropriate modifications, our results apply also to ℓ-adic sheaves, or any other reasonable sheaf-theoretic situation.

Much of the motivation for the study of sheaves on Bun_G comes from the so-called *geometric Langlands program*. In line with this, the main results of this paper have a transparent meaning in terms of this program, see Sect. 0.2. However, one can also view them from the perspective of the classical theory of automorphic functions (rather, we will see phenomena that so far have not been studied classically).

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0.1.2. *Constant term and Eisenstein series functors.* – To explain what is done in this paper we will first recall the main result of [6].

Let $P \subset G$ be a parabolic subgroup with Levi quotient M. The diagram of groups

$$G \hookleftarrow P \twoheadrightarrow M$$

gives rise to a diagram of stacks

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "\mathsf{p}"'] \arrow[dr, "\mathsf{q}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_M . \end{tikzcd} \tag{0.1}$$

Using this diagram as "pull-push," one can write down several functors connecting the categories of D-modules on Bun_G and Bun_M, respectively. By analogy with the classical theory of automorphic functions, we call the functors going from Bun_M to Bun_G "Eisenstein series," and the functors going from Bun_G to Bun_M "constant term".

Namely, we have

$$\begin{aligned} \mathrm{Eis}_! &:= \mathsf{p}_! \circ \mathsf{q}^*, \quad \text{D-mod}(\mathrm{Bun}_M) \to \text{D-mod}(\mathrm{Bun}_G), \\ \mathrm{Eis}_* &:= \mathsf{p}_* \circ \mathsf{q}^!, \quad \text{D-mod}(\mathrm{Bun}_M) \to \text{D-mod}(\mathrm{Bun}_G), \\ \mathrm{CT}_! &:= \mathsf{q}_! \circ \mathsf{p}^*, \quad \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M), \\ \mathrm{CT}_* &:= \mathsf{q}_* \circ \mathsf{p}^!, \quad \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M). \end{aligned}$$

Note that unlike the classical theory, where there is only one pull-back and one push-forward for functions, for sheaves there are two options: ! and $*$, for both pull-back and push-forward. The interaction of these two options is one way to look at what this paper is about.

Among the above functors, there are some obvious adjoint pairs: $\mathrm{Eis}_!$ is the left adjoint of CT_*, and $\mathrm{CT}_!$ is the left adjoint of Eis_*.

In addition to this, the following, perhaps a little unexpected, result was proved in [6]:

Theorem 0.1.3. – *The functors* $\mathrm{CT}_!$ *and* CT_*^- *are canonically isomorphic.*

In the statement of the theorem the superscript "–" means the constant term functor taken with respect to the *opposite* parabolic P^- (note that the Levi quotients of P and P^- are canonically identified).

Our goal in the present paper is to understand what implication the above-mentioned isomorphism

$$\mathrm{CT}_! \simeq \mathrm{CT}_*^-$$

has for the Eisenstein series functors $\mathrm{Eis}_!$ and Eis_*. The conclusion will be what we will call a "strange" functional Equation (0.9), explained below.

In order to explain what the "strange" functional equation does, we will need to go a little deeper into what one may call the "functional-analytic" aspects of the study of Bun_G.

0.1.4. *Verdier duality on stacks*. – The starting point for the "analytic" issues that we will be dealing with is that the stack Bun_G is *not quasi-compact* (this is parallel to the fact that in the classical theory, the automorphic space is not compact, leading to a host of interesting analytic phenomena). The particular phenomenon that we will focus on is the absence of the usual Verdier duality functor, and what replaces it.

First off, it is well-known (see, e.g., [5, Sect. 2]) that if $\mathcal{Y}$ is an arbitrary reasonable (1) quasi-compact algebraic stack, then the category $\text{D-mod}(\mathcal{Y})$ is compactly generated and naturally self-dual.

Perhaps, the shortest way to understand the meaning of self-duality is that the subcategory $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}(\mathcal{Y})$ consisting of compact objects carries a canonically defined contravariant self-equivalence, called Verdier duality. A more flexible way of interpreting the same phenomenon is an equivalence, denoted $\mathbf{D}_{\mathcal{Y}}$, between $\text{D-mod}(\mathcal{Y})$ and its *dual* category $\text{D-mod}(\mathcal{Y})^\vee$ (we refer the reader to [4, Sect. 1], where the basics of the notion of duality for DG categories are reviewed).

Let us now remove the assumption that $\mathcal{Y}$ be quasi-compact. Then there is another geometric condition, called "truncatability" that ensures that $\text{D-mod}(\mathcal{Y})$ is compactly generated (see [5, Definition 4.1.1], where this notion is introduced). We remark here that the goal of the paper [5] was to show that the stack Bun_G is truncatable. The reader who is not familiar with this notion is advised to ignore it on the first pass.

Thus, let us assume that $\mathcal{Y}$ is truncatable. However, there still is no obvious replacement for Verdier duality: extending the quasi-compact case, one can define a functor

$$(\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}} \to \text{D-mod}(\mathcal{Y}),$$

but it no longer lands in $\text{D-mod}(\mathcal{Y})^c$ (unless $\mathcal{Y}$ is a disjoint union of quasi-compact stacks). In the language of dual categories, we have a functor

$$\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y}),$$

but it is no longer an equivalence. (2)

In particular, the functor $\text{Ps-Id}_{\mathrm{Bun}_G,\mathrm{naive}}$ is *not* an equivalence, unless G is a torus.

0.1.5. *The pseudo-identity functor*. – To potentially remedy this, V. Drinfeld suggested another functor, denoted

$$\text{Ps-Id}_{\mathcal{Y},!} : \text{D-mod}(\mathcal{Y})^\vee \to \text{D-mod}(\mathcal{Y}),$$

see [5, Sect. 4.4.8] or Sect. 3.1 of the present paper.

Now, it is not true that for all truncatable stacks $\mathcal{Y}$, the functor $\text{Ps-Id}_{\mathcal{Y},!}$ is an equivalence. In [5] the stacks for which it is an equivalence are called "miraculous".

We can now formulate the main result of this paper (conjectured by V. Drinfeld):

THEOREM 0.1.6. – *The stack* Bun_G *is miraculous.*

(1) The word "reasonable" here does not have a technical meaning; the technical term is "QCA," which means that the automorphism group of any field-valued point is affine.

(2) The category $\text{D-mod}(\mathcal{Y})^\vee$ and the functor $\text{Ps-Id}_{\mathcal{Y},\mathrm{naive}}$ will be described explicitly in Sect. 1.2.