ISSN 0012-9593

quatrième série - tome 50 fascicule 5 septembre-octobre 2017

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Goulwen FICHOU & Masahiro SHIOTA

Real Milnor Fibres and Puiseux Series

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 50, 2017, p. 1205 à 1240

REAL MILNOR FIBRES AND PUISEUX SERIES

BY GOULWEN FICHOU AND MASAHIRO SHIOTA

ABSTRACT. – Given a real polynomial function and a point in its zero locus, we defined a set consisting of algebraic real Puiseux series naturally attached to these data. We prove that this set determines the topology and the geometry of the real Milnor fibre of the function at this point. To achieve this goal, we balance between the tameness properties of this set of Puiseux series, considered as a real algebraic object over the field of algebraic Puiseux series, and its behavior as an infinite dimensional object over the real numbers.

RÉSUMÉ. – On associe à un zéro d'une fonction polynomiale réelle, un ensemble de séries de Puiseux algébriques. Cet ensemble détermine à la fois la topologie et la géométrie des fibres de Milnor de la fonction en ce point. Pour démontrer cela, on combine les propriétés de modération de cet ensemble de séries de Puiseux, considéré comme un objet de la géométrie réelle sur le corps des séries de Puiseux algébriques, avec son comportement en tant qu'espace de dimension infini sur le corps des nombres réels.

Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{N}$ denote the non-negative integers. Denote by $\tilde{\mathbb{R}}$ the field of continuous semialgebraic curve germs $\gamma : (0, \varepsilon) \to \mathbb{R}$ at 0, which we identify with the field $\tilde{\mathbb{R}} = \mathbb{R}_{alg}((t^{\mathbb{Q}}))$ of algebraic Puiseux series over $\mathbb{R}$ (cf. [1]). Recall that the subring of algebraic Puiseux series of the form $\sum_{i\in\mathbb{N}} a_i t^{i/p}$ is carried to the subring of continuous semialgebraic curve germs $[0, \infty) \to \mathbb{R}$ at 0.

Let f be a polynomial function on $\mathbb{R}^n$, with $n \in \mathbb{N}^*$, and denote by $\tilde{f} : \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ the extension of f defined by $\tilde{f}(\gamma(t)) = f \circ \gamma(t)$ for $\gamma \in \tilde{\mathbb{R}}$ an algebraic Puiseux series. Let $x_0 \in \mathbb{R}^n$ be a vanishing point for f. The object of study of the present paper is the subset $\mathcal{F}_{f,x_0} \subset \tilde{\mathbb{R}}^n$ of continuous semialgebraic curve germs $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, x_0)$ such that $f \circ \gamma(t) = t$, namely

$$\mathcal{F}_{f,x_0} = \{\gamma \in \tilde{\mathbb{R}}^n : \ \gamma(0) = x_0, \ \ f \circ \gamma(t) = t\}.$$

The first author has been supported by the ANR project DEFIGEO. The second author thanks M. Takase for his advice on knots.

 doi:10.24033/asens.2343

We aim to relate the topology and the geometry of $\mathcal{F}_{f,x_0}$ to the topology and the geometry of the Milnor fibre associated with the real polynomial function f at x_0, which can be described as the semialgebraic set

$$F_{r,a}(f) = \{x \in \mathbb{R}^n : \ |x - x_0| < r, \ \ f(x) = a\}$$

for $0 < a \ll r \ll 1$.

The local study of the singular points of complex hypersurfaces has a rich story initiated by J. Milnor in its fundamental book [15], where he established the so-called Milnor fibration Theorem. More recently, the development of Motivic Integration [6] has brought a new enlightenment on the subject, via the motivic zeta function introduced by J. Denef and F. Loeser, together with the analytic Milnor fibre defined by J. Nicaise and J. Sebag [16]. Even more recently, E. Hrushovski and F. Loeser have established a direct connection between the analytic Milnor fibre and the topological Milnor fibre, together with the motivic Milnor fibre [10], passing through the integration into valued fields developed by E. Hrushovski and D. Kazhdan [9].

In the real context, the action of the monodromy operator on the Milnor fibre disappears, which gives rise to the notion of positive and negative Milnor fibres, as studied for example by C. McCrory and A. Parusiński [13]. However the motivic counterpart, initiated by G. Comte and G. Fichou [2] and Y. Yin [22], does not provide yet a full understanding of the global feature. Our motivation to consider the set $\mathcal{F}_{f,x_0}$ is to study a naive real version of the analytic Milnor fibre. Looking at points, we obtain a set of real algebraic Puiseux series, which can almost (or more precisely weakly, see below) be considered as a classical semialgebraic object in real algebraic geometry, replacing the field of real numbers by the real closed field of algebraic Puiseux series over $\mathbb{R}$. Note that, if the definition of $\mathcal{F}_{f,x_0}$ makes sense for any continuous semialgebraic function f, however $\mathcal{F}_{f,x_0}$ is not necessarily a semialgebraic subset of $\tilde{\mathbb{R}}^n$ due to the condition that the arcs considered have their origin in $x_0 \in \mathbb{R}^n$. This condition can be described by a valuation condition, saying that we consider those arcs with strictly positive valuation after the translation by $\gamma \mapsto \gamma - x_0$. Such sets are sometimes called T-convex, or weakly o-minimal [12].

Our aim in this paper is to show that the topology and the geometry of $\mathcal{F}_{f,x_0}$, which is definitively a natural and intrinsic object in real geometry, determine the topology and the geometry of the (positive) real Milnor fibre of f at x_0, which can be considered as a semialgebraic set, but only well-defined up to the choice of (sufficiently small) constants. Note that a similar result holds for the negative Milnor fibre, a negative sign in front of t being necessary in the definition of $\mathcal{F}_{f,x_0}$ in that case. We prove also that some natural homology groups of $\mathcal{F}_{f,x_0}$ coincide with the classical homology groups of the (positive) real Milnor fibre. The main achievement of the paper states that $\mathcal{F}_{f,x_0}$ completely determines the (positive) real Milnor fibre up to semialgebraic homeomorphism (cf Theorem 4.1.(1)), and determines it up to Nash diffeomorphism in dimensions different from 5 and 6 (cf Theorem 4.1.(2)).

As a real closed field, $\tilde{\mathbb{R}}$ is naturally equipped with the ordered topology which coincides with the t-adic topology. But we may also regard $\tilde{\mathbb{R}}$ as a $\mathbb{R}$-vector subspace of $\mathbb{R}^{\mathbb{Q}} = \prod_{r \in \mathbb{Q}} \mathbb{R}\, t^r$, and consider $\mathbb{R}^{\mathbb{Q}}$ as a topological space with the product topology. Then $\tilde{\mathbb{R}}$ may also be equipped with the induced topology.

In the paper, we balance between the tameness properties of $\mathcal{F}_{f,x_0}$ as a subset of $\tilde{\mathbb{R}}^n$ close to be semialgebraic (up to the valuation condition), and its behavior as a topological set of infinite dimension in $\mathbb{R}^{\mathbb{Q}}$. This situation leads to the study in the first part of the paper of the notion of weak continuity, and its relationship with the continuity of semialgebraic maps defined over $\tilde{\mathbb{R}}$. We discuss in part 2 the associated homology theories, preparing the material for the comparison of homologies given in the third part as Theorem 3.6. In the fourth part, we focus on the semialgebraic characterization of the real Milnor fibre, using notion from piecewise linear topology [17]. The last part is dedicated to the Nash characterization, using the theory of topological microbundles of J. Milnor [14].

Note that all the results in the paper work verbatim for a Nash function in place of a polynomial function f, contrary to the case of a real analytic function for the reason that a globally subanalytic triviality theorem, analog to the Nash triviality theorem in [4], is not yet available.

We denote by $\mathbb{R}$ the field of real numbers, and by $\tilde{\mathbb{R}}$ the field of real Puiseux series in one variable that are algebraic over $\mathbb{R}[X]$. The field $\tilde{\mathbb{R}}$ is a real closed field, a non zero element $\gamma \in \tilde{\mathbb{R}}$ is given (uniquely if we impose that q is the smallest common denominator of the exponents) by

$$\gamma(t) = \sum_{i \geq p} a_i t^{i/q}, \quad a_i \in \mathbb{R},\ a_p \neq 0,\ (p,q) \in \mathbb{Z} \times \mathbb{N}^*,$$

and γ is positive if $a_p > 0$.

To distinguish an interval (a, b) in $\mathbb{R}$ and in $\tilde{\mathbb{R}}$, we write (a, b) in $\tilde{\mathbb{R}}$ as $(a, b)_{\tilde{\mathbb{R}}}$. A semialgebraic set is a semialgebraic set over $\mathbb{R}$ and an $\tilde{\mathbb{R}}$-semialgebraic set is a semialgebraic over $\tilde{\mathbb{R}}$. For a semialgebraic set $X \subset \mathbb{R}^n$, denote by $\tilde{X}$ the set of germs at $0 \in \mathbb{R}$ of continuous semialgebraic functions from $(0;\infty)$ to X. For $x \in X$, we denote $\tilde{x} \in \tilde{X}$ the germ of the constant function equal to x. Let X and Y be semialgebraic sets and $h : X \to Y$ be a continuous semialgebraic map. Let $\tilde{h} : \tilde{X} \to \tilde{Y}$ be defined by $\tilde{h}(\gamma(t)) = h \circ \gamma(t)$ for $\gamma \in \tilde{\mathbb{R}}$. We know from [8] that $\tilde{h}$ is continuous for the t-adic topology, but not necessarily for the product topology.

We define $\mathfrak{m}_+ \subset \tilde{\mathbb{R}}$ to be the set of infinitely small positive elements in $\tilde{\mathbb{R}}$, namely:

$$\mathfrak{m}_+ = \{\gamma \in \tilde{\mathbb{R}} : \ \ 0 < \gamma < \tilde{x} \text{ for all } x \in (0, +\infty)\}.$$

1. Weak continuity

A *Nash manifold* is a semialgebraic C^∞-submanifold of some $\mathbb{R}^n$ and a *Nash map* between Nash manifolds is a C^∞-map with semialgebraic graph. The Milnor fibre $F_f(r, a)$ as considered in the introduction is a Nash manifold with boundary, however the set $\mathcal{F}_{f,x_0}$ is not so, even by changing $\mathbb{R}$ with the field $\tilde{\mathbb{R}}$ of algebraic Puiseux series over $\mathbb{R}$. We will regard $\mathcal{F}_f$ as a local $\tilde{\mathbb{R}}$-Nash manifold, see Definition 1.1 below. Properties of semialgebraic sets, Nash manifolds and Nash maps are explained in [1] and [18]. We will recall some of them for the convenience of the reader.

For any ordered field R, we define a topology on R by considering as open sets the open intervals, and we called it the R-topology. We denote by $(a, b)_R$ the open interval defined by a and b in R, and called it an R-interval in order to emphasize again the dependence on R (if R is not the field of real numbers). In the same way as in the real number case, we