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MOTIVES OF RIGID ANALYTIC TUBES AND NEARBY MOTIVIC SHEAVES

BY JOSEPH AYOUB, FLORIAN IVORRA AND JULIEN SEBAG

ABSTRACT. – Let k be a field of characteristic zero, $R = k[[t]]$ the ring of formal power series and $K = k((t))$ its fraction field. Let X be a finite type R-scheme with smooth generic fiber. Let $\mathscr{X}$ be the t-adic completion of X and $\mathscr{X}_\eta$ the generic fiber of $\mathscr{X}$. Let $Z \subset X_\sigma$ be a locally closed subset of the special fiber of X. In this article, we establish a relation between the rigid motive of $]Z[$ (the tube of Z in $\mathscr{X}_\eta$) and the restriction to Z of the nearby motivic sheaf associated with the R-scheme X. Our main result, Theorem 7.1, can be interpreted as a motivic analog of a theorem of Berkovich.

As an application, given a rational point $x \in X_\sigma$, we obtain an equality, in a suitable Grothendieck ring of motives, between the motivic Milnor fiber of Denef-Loeser at x and the class of the rigid motive of the analytic Milnor fiber of Nicaise-Sebag at x.

RÉSUMÉ. – Soient k un corps de caractéristique nulle, $R = k[[t]]$ l'anneau des séries formelles sur k et $K = k((t))$ son corps des fractions. Soit X un R-schéma de type fini génériquement lisse. Soient $\mathscr{X}$ la complétion t-adique de X et $\mathscr{X}_\eta$ sa fibre générique. Soit $Z \subset X_\sigma$ un sous-ensemble localement fermé de X. Dans cet article, nous lions le motif rigide du tube $]Z[$ de Z dans $\mathscr{X}_\eta$ à la restriction à Z du faisceau cycles proches motivique associé au R-schéma X. Le théorème 7.1, qui est notre résultat principal, peut être interprété comme un analogue motivique d'un théorème de Berkovich.

Comme application, étant donné un point rationnel $x \in X_\sigma$, nous obtenons une égalité dans un anneau de Grothendieck de motifs adéquat entre la fibre de Milnor motivique de Denef-Loeser en x et la classe du motif rigide de la fibre de Milnor analytique de Nicaise-Sebag en x.

1. Introduction

1.1. – Let k be a field of characteristic zero, $R = k[[t]]$ be the ring of formal power series and $K = k((t))$ be its fraction field. Let Λ be a commutative ring (that we call the ring of coefficients). While the main body of the article is written in a greater generality, we restrict

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ourselves in the introduction to the categories of motives without transfers $\mathbf{DA}(k,\Lambda)$ and its rigid analytic version $\mathbf{RigDA}(K,\Lambda)$. These categories are related by triangulated functors

$$\mathbf{RigDA}(K,\Lambda) \xrightarrow{\mathfrak{R}} \mathbf{QUDA}(k,\Lambda) \xrightarrow{1^*} \mathbf{DA}(k,\Lambda),$$

where $\mathbf{QUDA}(k,\Lambda)$ is the full triangulated subcategory of $\mathbf{DA}(\mathbf{G}_{m,k},\Lambda)$ whose objects are the quasi-unipotent motives; the functor $\mathfrak{R}$ is an equivalence of categories (see [6, Scholie 1.3.26]) and 1^* is the pullback functor along the unit section. For a quick recollection on motives and rigid motives, the reader is referred to §3.

1.2. – Let X be a finite type R-scheme and denote by $f\colon X \to \mathsf{Spec}(R)$ its structural morphism. We denote by X_η and X_σ the generic and special fibers of X.

By [3, Chapitre 3] (see also [6, §A.1]), one has the nearby motivic sheaf $\Psi_f(\Lambda_{X_\eta})$ associated with f; this is an object of $\mathbf{DA}(X_\sigma,\Lambda)$. It realizes to the classical complexes of nearby cycles by [4, Théorème 4.9] (for the Betti realization and when X is the base-change of a finite type $k[t]$-scheme) and [5, Théorème 10.11] (for the ℓ-adic realization).

Consider the t-adic completion $\hat{f} : \mathscr{X} \to \mathsf{Spf}(R)$ of f and denote by $\mathscr{X}_\eta$ the generic fiber of $\mathscr{X}$. The rigid analytic variety $\mathscr{X}_\eta$ is an open analytic subvariety of the analytification X_η^{an} of the algebraic generic fiber X_η (e.g., see [12, (0.3.5)]). Given a locally closed subset $Z \subset X_\sigma$ (endowed with its reduced structure), denote by $]Z[$ its tube; this is an open rigid analytic subvariety of $\mathscr{X}_\eta$.

Assume that the rigid analytic variety $\mathscr{X}_\eta$ is smooth over K; this is the case for instance if the scheme X_η is smooth over K. Let $\mathrm{M}^\vee_{\mathrm{rig}}(]Z[)$ be the cohomological motive of $]Z[$; this is an object of $\mathbf{RigDA}(K,\Lambda)$. The main theorem of this article is the following (see Theorem 7.1 for a more general statement):

THEOREM. – *Denote by $z : Z \hookrightarrow X_\sigma$ the inclusion. Then, there is a canonical isomorphism*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(]Z[)) \simeq (f_\sigma)_* z_* z^* \Psi_f(\Lambda_{X_\eta}) \tag{1}$$

in the category of motives $\mathbf{DA}(k,\Lambda)$.

Taking $Z = X_\sigma$, one gets that the cohomological motive $\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{X}_\eta)$ is related to the nearby motivic sheaf by a canonical isomorphism

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{X}_\eta)) \simeq (f_\sigma)_* \Psi_f(\Lambda_{X_\eta})$$

in $\mathbf{DA}(k,\Lambda)$. In fact, we first prove this particular case of our main theorem (see Theorem 4.11 and Corollary 4.12) and then use it, with other ingredients, to derive the general case.

As a by-product of this work, we show that the rigid motives of tubes are compact (see Proposition 5.9), and we extend to motivic stable homotopy theory the computation of nearby motivic sheaves obtained previously by Ayoub in the context of étale motives (see Theorem 6.1).

1.3. – Our main theorem is a motivic analog of a theorem of Berkovich that we explain now. Let $\overline{K}$ be the completion of an algebraic closure of the valued field K and let $\overline{k}$ be its residue field. Set $\overline{Z} = Z \times_k \overline{k}$ and $]\overline{Z}[\, =]Z[\hat{\times}_K \overline{K}$. In [10, 11], Berkovich constructed canonical isomorphisms of étale hypercohomology groups

$$\mathrm{H}^i_{\text{ét}}(]\overline{Z}[, \mathbf{Q}_\ell) \simeq \mathbb{H}^i_{\text{ét}}(\overline{Z}, R\Psi_{\hat{f}}(\mathbf{Q}_{\ell,\mathscr{X}_\eta})_{|\overline{Z}}) \simeq \mathbb{H}^i_{\text{ét}}(\overline{Z}, R\Psi_f(\mathbf{Q}_{\ell,X_\eta})_{|\overline{Z}}). \tag{2}$$

(Here the tube $]Z[$ has to be considered as a Berkovich space in order to take its non-archimedean étale cohomology [9].) The first isomorphism is shown in [11, Corollary 3.5]; the second one follows from [10, Corollary 5.3].

We expect that the isomorphism (1) realizes to the composition of the isomorphisms in (2). However, we do not make any attempt to check this in this article. It is worth noting that Berkovich's theorem holds over general non-archimedean fields whereas, for the very statement of our theorem, we need to assume that K has equal characteristic zero. Indeed, this is required for [6, Scholie 1.3.26] which ensures the existence of the equivalence $\mathfrak{R}$.

1.4. – Let $x \in X_\sigma$ be a rational point. In [16, Définition 4.2.1], Denef and Loeser have introduced the *motivic Milnor fiber* $\psi_{f,x} \in \mathscr{M}_k$ as the limit of the motivic zeta function associated with f; in [33], Nicaise and Sebag have defined the analytic Milnor fiber at x to be $\mathscr{F}_x =]x[$. The present work and [24] show that the (stable) motivic homotopy theory is a natural framework to relate and study these different notions of Milnor fiber. A particular case of our main theorem (see Theorem 8.8) gives an isomorphism of motives

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{F}_x)) \simeq x^*\Psi_f(\Lambda_{X_\eta}).$$

Theorem 6.1 shows that [24, Theorem 1.2] remains valid in a more general setting, and we deduce the following formula in the Grothendieck group of constructible motives

$$[1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{F}_x))] = \chi_{k,\mathrm{c}}(\psi_{f,x}). \tag{3}$$

Here, we denote by $\chi_{k,\mathrm{c}} : \mathscr{M}_k \to \mathrm{K}_0(\mathbf{DA}_{\mathrm{ct}}(k,\Lambda))$ the motivic Euler characteristic [24, Lemma 2.1].

Formula (3) expresses the fact that the motivic Milnor fiber of Denef-Loeser, at least as a class in the Grothendieck ring of constructible motives, is determined by the rigid motive of the analytic Milnor fiber. A formula of a similar nature, comparing the motivic Milnor fiber of Denef-Loeser to the analytic Milnor fiber, appears in [23, Corollary 8.4.2]. (See Remarks 8.14 and 8.15 for an attempt to relate the two formulas.)

Notations, conventions

1.5. – Although this is not really necessary, all schemes, formal schemes and rigid varieties will be assumed to be *separated*. Schemes and formal schemes will be also assumed to be *quasi-compact*. Given a base-scheme S, we denote by Sm/S the category of smooth S-schemes. Given a non-archimedean complete field F, we denote by SmRig/F the category of smooth rigid F-varieties.

When there is no risk of confusion, a scheme S will be identified with its maximal reduced subscheme that we denote by S_{red}. Also, a locally closed subset of a scheme will be automatically endowed with its reduced subscheme structure. The same applies for rigid analytic varieties.