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# FROM A KAC-LIKE PARTICLE SYSTEM TO THE LANDAU EQUATION FOR HARD POTENTIALS AND MAXWELL MOLECULES

BY NICOLAS FOURNIER AND ARNAUD GUILLIN

ABSTRACT. – We prove a quantitative result of convergence of a conservative stochastic particle system to the solution of the homogeneous Landau equation for hard potentials. There are two main difficulties: (i) the conservativeness of the particle system is an obstacle for approximate independence, as is the case for *true* physical particle systems; (ii) the known stability results for this class of Landau equations concern *regular* solutions and seem difficult to extend to study the rate of convergence of some empirical measures. Due to (i), we have to use a double-coupling. We first couple our particle system with some *non independent* nonlinear processes, of which the law solves, in some sense, the Landau equation. We then introduce a second coupling to prove that these nonlinear processes are not so far from being independent. To overcome (ii), we prove a new stability result for the Landau equation for hard potentials concerning very general measure solutions. Using finally some ideas of Rousset [26], we show that in the case of Maxwell molecules, the convergence of the particle system is uniform in time.

RÉSUMÉ. – Nous prouvons des résultats quantitatifs de convergence d'un système de particules conservatif vers la solution de l'équation de Landau homogène pour des potentiels durs. Il y a deux principales difficultés : (i) le caractère conservatif du système est un obstacle pour obtenir de l'indépendance (même approchée), comme c'est le cas pour de *vrais* systèmes de particules physiques ; (ii) les résultats connus de stabilité pour ces équations de Landau concernent des solutions régulières et paraissent difficiles à étendre pour étudier la vitesse de convergence de mesures empiriques. Pour le point (i), nous procédons à un double couplage. Nous couplons d'abord notre système avec des processus non linéaires *non indépendants* dont la loi résout en un certain sens l'équation de Landau. Nous construisons ensuite un second couplage afin de montrer que ces processus non linéaires ne sont pas loin d'être indépendants. Pour résoudre (ii), nous établissons de nouveaux résultats de stabilité pour l'équation de Landau pour des potentiels durs et des solutions de type mesure très générales. Finalement, en utilisant des idées de Rousset [26], nous montrons que dans le cas des molécules maxwelliennes, la convergence du système de particules est uniforme en temps.

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## 1. Introduction and main results

### 1.1. The Landau equation

The homogeneous Landau equation reads

$$\partial_t f_t(v) = \frac{1}{2}\mathrm{div}_v\Big(\int_{\mathbb{R}^3} a(v-v_*)[f_t(v_*)\nabla f_t(v) - f_t(v)\nabla f_t(v_*)]\,dv_*\Big). \tag{1}$$

The unknown $f_t : \mathbb{R}^3 \mapsto \mathbb{R}$ stands for the velocity-distribution in a plasma and the initial condition $f_0$ is given. We denote by $S_3^+$ the set of symmetric nonnegative $3\times 3$ matrices. The function $a : \mathbb{R}^3 \mapsto S_3^+$ is given, for some $\gamma \in [-3,1]$, by

$$a(v) = |v|^{2+\gamma}\Pi_{v^\perp}, \quad \text{where} \quad \Pi_{v^\perp} = \mathbf{I}_3 - \frac{v\otimes v}{|v|^2}$$

is the projection matrix onto $v^\perp$. The only physically relevant case, namely $\gamma = -3$ which corresponds to a Coulomb interaction, is unfortunately the most difficult. For mathematical results, let us mention the recent papers by Desvillettes [9] and Carrapatoso, Desvillettes and He [7] and the references therein. The other cases are interesting mathematically and numerically. In particular, the Landau equation can be seen as an approximation of the Boltzmann equation in the asymptotic of *grazing collisions*, as rigorously shown by Villani [31] for all values of $\gamma \in [-3,1]$. We are concerned here with Maxwell molecules ($\gamma = 0$) and hard potentials ($\gamma \in (0,1]$). The well-posedeness, regularization properties and large-time behavior of the Landau equation have been studied in great details by Villani [32] for Maxwell molecules and by Desvillettes and Villani [10, 11] for hard potentials. We finally refer to the long reviews paper of Villani [33] and Alexandre [1] on the Boltzmann and Landau models.

### 1.2. Notation

We denote by $\mathcal{P}(\mathbb{R}^3)$ the set of probability measures on $\mathbb{R}^3$. When $f \in \mathcal{P}(\mathbb{R}^3)$ has a density, we also denote by $f \in L^1(\mathbb{R}^3)$ this density.

For $q > 0$, we set $\mathcal{P}_q(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty\}$, where $m_q(f) = \int_{\mathbb{R}^3} |v|^q f(dv) < \infty$. For $\alpha > 0$ and $f \in \mathcal{P}(\mathbb{R}^3)$, we put $\mathcal{E}_\alpha(f) = \int_{\mathbb{R}^3} \exp(|v|^\alpha) f(dv)$. The entropy of $f \in \mathcal{P}(\mathbb{R}^3)$ is defined by $H(f) = \int_{\mathbb{R}^3} f(v)\log f(v)dv$ if $f$ has a density and by $H(f) = \infty$ else.

We will use the Wasserstein distance defined as follows. For $f, g \in \mathcal{P}_2(\mathbb{R}^3)$, we introduce $\mathcal{H}(f,g) = \{R \in \mathcal{P}(\mathbb{R}^3\times\mathbb{R}^3) : R \text{ has marginals } f \text{ and } g\}$ and we set

$$\mathcal{W}_2(f,g) = \inf\Big\{\Big(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-w|^2 R(dv,dw)\Big)^{1/2} : R \in \mathcal{H}(f,g)\Big\}.$$

See Villani [34] for many details on this distance.

We also define, for $v \in \mathbb{R}^3$,

$$b(v) = \mathrm{div}\, a(v) = -2|v|^\gamma v \quad \text{and} \quad \sigma(v) = [a(v)]^{1/2} = |v|^{1+\gamma/2}\Pi_{v\perp}.$$

For $f \in \mathcal{P}(\mathbb{R}^3)$ and $v \in \mathbb{R}^3$, we set

$$b(f,v) := \int_{\mathbb{R}^3} b(v-v_*)f(dv_*), \quad a(f,v) := \int_{\mathbb{R}^3} a(v-v_*)f(dv_*), \quad a^{1/2}(f,v) := \Big[a(f,v)\Big]^{1/2}.$$

More generally, we will write $\varphi(f,v) = \int_{\mathbb{R}^3} \varphi(v-v_*)f(dv_*)$ when $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$. We emphasize that $a^{1/2}(f,v)$ is $[a(f,v)]^{1/2}$ and is not $\int_{\mathbb{R}^3} a^{1/2}(v-v_*)f(dv_*)$.

Finally, for $A$ and $B$ two $3\times 3$ matrices, we put $\|A\|^2 = \mathrm{Tr}(AA^*)$ and $\langle\langle A, B\rangle\rangle = \mathrm{Tr}(AB^*)$.

### 1.3. Well-posedness

We will use the following notion of weak solutions.

DEFINITION 1. – *Let $\gamma \in [0,1]$. We say that $f = (f_t)_{t\geq 0}$ is a weak solution to* (1) *if it belongs to $L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_{2+\gamma}(\mathbb{R}^3))$ and if for all $\varphi \in C^2_b(\mathbb{R}^3)$, all $t \geq 0$,*

$$\int_{\mathbb{R}^3} \varphi(v) f_t(dv) = \int_{\mathbb{R}^3} \varphi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L\varphi(v,v_*) f_s(dv) f_s(dv_*) ds, \tag{2}$$

*where*

$$L\varphi(v,v_*) := \frac{1}{2}\sum_{k,l=1}^3 a_{kl}(v-v_*)\partial^2_{kl}\varphi(v) + \sum_{k=1}^3 b_k(v-v_*)\partial_k\varphi(v).$$

*A weak solution $f$ is* conservative *if it conserves momentum and energy, that is $\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv)$ and $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$.*

An important remark is that $|L\varphi(v,v_*)| \leq C_\varphi(1+|v|+|v_*|)^{2+\gamma}$ for $\varphi \in C^2_b(\mathbb{R}^3)$ and since $f \in L^\infty_{\mathrm{loc}}([0,\infty), \mathcal{P}_{2+\gamma}(\mathbb{R}^3))$, every term makes sense in (2). Our first result concerns well-posedness and stability.

THEOREM 2. – *(i) If $\gamma = 0$, then for any $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$,* (1) *has a unique weak solution $f = (f_t)_{t\geq 0}$ starting from $f_0$. This solution is conservative. If moreover $H(f_0) < \infty$, then $H(f_t) \leq H(f_0)$ for all $t \geq 0$. If $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q > 2$, then $\sup_{[0,\infty)} m_q(f_t) < \infty$. Finally, for any other weak solution $g = (g_t)_{t\geq 0}$ to* (1)*, it holds that $\mathcal{W}_2(f_t, g_t) \leq \mathcal{W}_2(f_0, g_0)$ for all $t \geq 0$.*

*(ii) If $\gamma \in (0,1]$, consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with $\mathcal{E}_\alpha(f_0) < \infty$ for some $\alpha \in (\gamma, 2)$. Then* (1) *has a unique weak solution $f = (f_t)_{t\geq 0}$ starting from $f_0$. Moreover, this solution is conservative and $\sup_{t\geq 0}\mathcal{E}_\alpha(f_t) < \infty$. If $H(f_0) < \infty$, then $H(f_t) \leq H(f_0)$ for all $t \geq 0$. Finally, for all $\eta \in (0,1)$, all $T > 0$ and any other weak solution to $g = (g_t)_{t\geq 0}$ to* (1)*, it holds that $\sup_{[0,T]} \mathcal{W}_2(f_t, g_t) \leq C_{\eta,T}(\mathcal{W}_2(f_0,g_0))^{1-\eta}$, the constant $C_{\eta,T}$ depending only on $\eta, T, \gamma, \alpha$ and on (upper bounds of) $\mathcal{E}_\alpha(f_0)$ and $\sup_{[0,T]} m_{2+\gamma}(g_t)$.*

Point (i) is well-known, even if we found no precise reference for all the claims of the statement. The well-posedness, propagation of moments and entropy dissipation have been checked by Villani [32] when $f_0$ has a density and the well-posedness when $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ has been established by Guérin [20]. The noticeable fact that $\mathcal{W}_2$ decreases along solutions was discovered by Tanaka [30] for the Boltzmann equation for Maxwell molecules, see also Carrapatoso [6, Lemma 4.15].

Similarly, the existence part in point (ii) is more or less standard: the well-posedness, propagation of moments and entropy dissipation can be found in [10] when $H(f_0) < \infty$, but $H(f_0) < \infty$ is mainly assumed for simplicity. The propagation of exponential moments seems to be new, but far from surprising: it is well-known (and more complicated) for the Boltzmann equation for hard potentials, as was discovered by Bobylev [5], see also Alonso *et al.* [2].

On the contrary, the uniqueness/stability part in point (ii) seems to be new and rather interesting. As far as we know, the best available uniqueness result is the one of Desvillettes