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THE CANONICAL MEASURE ON A REDUCTIVE p-ADIC GROUP IS MOTIVIC

BY JULIA GORDON AND DAVID ROE

ABSTRACT. – Let G be a connected reductive group over a non-Archimedean local field. We prove that its parahoric subgroups are definable in the Denef-Pas language, which is a first-order language of logic used in the theory of motivic integration developed by Cluckers and Loeser. The main technical result is the definability of the connected component of the Néron model of a tamely ramified algebraic torus. As a corollary, we prove that the canonical Haar measure on G, which assigns volume 1 to the particular *canonical* maximal parahoric defined by Gross in [9], is motivic. This result resolves a technical difficulty that arose in [4] and [12, Appendix B] and permits a simplification of some of the proofs in those articles. It also allows us to show that formal degree of a compactly induced representation is a motivic function of the parameters defining the representation.

RÉSUMÉ. – Soit G un groupe algébrique réductif connexe sur un corps local non-archimédien. Dans ce papier, nous démontrons que ses sous-groupes parahoriques sont définissables dans le langage de Denef-Pas, qui est le langage utilisé dans la théorie d'intégration motivique de Cluckers et Loeser. Notre résultat principal établit que la composante connexe du modèle de Néron d'un tore modérément ramifié est définissable. Ce résultat implique que la mesure canonique sur G (qui donne le volume 1 à un sous-groupe parahorique spécifique, défini par Gross [9]), est une mesure motivique. Ce résultat permet une simplification de quelques preuves dans [4] et [12, Appendix B]. Finalement, nous montrons que le degré formel des représentations supercuspidales dans une famille paramétrique est une fonction motivique des paramètres.

1. Introduction

The goal of this paper is to complete a technical step in the definable formulation of the representation theory of p-adic groups, a project started by T.C. Hales in 1999. Here, the word "definable" is as in the theory of motivic integration developed by R. Cluckers and F. Loeser [7].

Specifically, we will prove that the parahoric subgroups of a connected reductive p-adic group are definable using the Denef-Pas language, which is the language used in the Cluckers-Loeser theory of motivic integration and its applications to representation theory of p-adic

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groups. As a consequence, we prove that the canonical Haar measure on a connected reductive group (which assigns the volume 1 to the canonical parahoric subgroup constructed by B. Gross [9]) is motivic.

For unramified groups, this statement has been known for a while [6]. For ramified groups, the definition of the canonical smooth model of **G** relies on the Néron model of a maximal torus in **G**, which does not behave well with respect to Galois descent. The main technical result of this paper is that the connected component of the Néron model of a tamely ramified torus is definable in the language of Denef-Pas. The difficulty in proving this result is caused by the fact that "taking the connected component" is not an operation that can be easily described by first-order logic.

This paper is split into two sections, the first leading up to Proposition 3, which shows that the connected component of the Néron model of a torus is definable, and the second giving applications to canonical measures and formal degrees.

We begin Section 2 by setting up notation and briefly reviewing the Denef-Pas language. In order to give formulas defining $\mathbf{T}(F)$ and $\mathscr{T}^{\circ}(\mathcal{O}_F)$ (where $\mathscr{T}^{\circ}$ denotes the connected component of the Néron model of the torus **T**) in this language, we need to parameterize the possible tori **T**. In Section 2.1, we describe the choices that can be made without reference to variables in F, such as fixing an abstract Galois group Γ and a lattice with action of Γ which will play the role of the cocharacter lattice of **T**. Section 2.2 then parameterizes field extensions with Galois group Γ, resulting in a parameterization of tori over F. Finally, in Section 2.3 we show that $\mathscr{T}^{\circ}(\mathcal{O}_F)$ is a definable subgroup of $\mathbf{T}(F)$. In Section 3 we prove two easy corollaries mentioned above, namely, that the canonical measure is motivic, and in a definable family of compactly-induced irreducible representations, formal degree is motivic.

Acknowledgment. – We thank Loren Spice for the discussion of formal degree, and the referee for multiple helpful suggestions and corrections.

2. Tori

We will use the notions of definable sets and definable functions, which will always refer to the Denef-Pas language. Formulas in the Denef-Pas language can have variables of three *sorts*: valued field (which will be denoted by *VF*), residue field (denoted by *RF*) and the value group. Even though we will often be working with ramified extensions, we always start with a local field F with normalized valuation, so the value group is $\mathbb{Z}$ (the *VF*-variables will range over F, and so their valuations will be in $\mathbb{Z}$). Formulas in the Denef-Pas language can be interpreted given a choice of a valued field *together with a uniformizer*. We refer the reader to [5] and references therein for the definitions of the Denef-Pas language, definable sets, etc.

For us, F will always be a non-Archimedean local field: either $\mathbb{F}_q(\!(t)\!)$ or a finite extension of $\mathbb{Q}_p$. As a consequence of the definition of a definable set, all statements in this paper will hold for any F of sufficiently large residue characteristic p, though we will give no effective bound on p. Given an integer $M > 0$, we will denote by Loc_M the collection of non-Archimedean local fields with residue characteristic greater than M.

For a local field F, we will denote its ring of integers by $\mathcal{O}_F$, its residue field by k_F, and let $q_F = \#k_F$. The symbol ϖ or ϖ_F will stand for the uniformizer of the valuation on F. A formula in the Denef-Pas language with n free VF-variables, m free RF-variables, and r free $\mathbb{Z}$-variables defines a subset of $F^n \times k_F^m \times \mathbb{Z}^r$. We will denote the definable set $F^n \times k_F^m \times \mathbb{Z}^r$ itself by $VF^n \times RF^m \times \mathbb{Z}^r$. In earlier works on motivic integration this set was denoted by $h[n, m, r]$. We will talk about definable subsets of $VF^n \times RF^m \times \mathbb{Z}^r$, meaning the subsets defined by Denef-Pas formulas with the right number of free variables of each sort, as above. For a definable subset $X \subset VF^n \times RF^m \times \mathbb{Z}^r$, and given a local field F, we will denote by $X(F)$ the *specialization* of X in F, i.e., the subset of $F^n \times k_F^m \times \mathbb{Z}^r$ obtained by interpreting in F all the formulas defining the set X. (1)

We start by setting up the framework for working with tori in the Denef-Pas language, following [6], [4] and [8].

2.1. Fixed choices

As in [8, §2.1], we begin by outlining our *fixed choices*, which are made before writing any formulas in the Denef-Pas language. For each *fixed choice* (which will be completely field-independent), we will further describe a definable set of parameters (which will then be allowed to range over a valued field F, its residue field k_F or $\mathbb{Z}$), in such a way that each tuple of parameters gives rise to an algebraic torus defined over F, and all isomorphism classes of algebraic F-tori arise via this construction.

We fix a finite group Γ and a normal subgroup $I \trianglelefteq \Gamma$, as well as enumerations of their elements $\Gamma = \{\sigma_1, \dots, \sigma_m\}$ and $I = \{\sigma_1, \dots, \sigma_e\}$. We make the convention that $\sigma_1 = 1$ and σ_m generates (2) Γ/I. When we eventually construct a torus $\mathbf{T}$ from the fixed choices and corresponding parameters, these groups will play the roles of $\mathrm{Gal}(E/F)$ and its inertia subgroup, where E is the splitting field of $\mathbf{T}$.

In order to define a torus $\mathbf{T}$, we will use the equivalence of categories between F-tori and free $\mathbb{Z}$-modules with a Galois action. To this end, we fix a positive integer n and an injective homomorphism

$$\theta : \Gamma \hookrightarrow \mathbf{GL}_n(\mathbb{Z}), \tag{1}$$

which gives $\mathbb{Z}^n$ an action of Γ. The Γ-module X defined by θ will play the role of $X_*(\mathbf{T})$.

Finally, we fix a resolution of X by an induced Γ-module Y, i.e., a surjective map $Y \to X$ where Y has a basis permuted by Γ (cf. [3, Satz 0.4.4]). To specify Y, we just give the matrix for the map $Y \to X$ of free abelian groups, together with the matrices giving $\gamma : Y \to Y$ for $\gamma \in \Gamma$. This resolution will allow us to definably cut out the connected component of the Néron model inside $\mathbf{T}(F)$.

2.2. Parameterizing field extensions and tori

We encode field extensions in the same way as in [4]. Namely, we parameterize Galois extensions E/F with $\mathrm{Gal}(E/F) \cong \Gamma$ and realize all tori over F that split over E with cocharacter lattice X. This parameterizes such tori as members of a family of definable sets, for all F of sufficiently large residue characteristic.

(1) This is the notation used in [8]; note that traditionally in the motivic integration literature, the specialization of a definable set X was denoted by X_F, but this notation generates too many subscripts for us.
(2) Note that we do not assume that σ_m is the Frobenius element, since p is not fixed.