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ECALLE'S ARBORIFICATION-COARBORIFICATION TRANSFORMS AND CONNES-KREIMER HOPF ALGEBRA

BY FRÉDÉRIC FAUVET AND FRÉDÉRIC MENOUS

ABSTRACT. – We give a natural and complete description of Ecalle's mould-comould formalism within a Hopf-algebraic framework. The arborification transform thus appears as a factorization of characters, involving the shuffle or quasishuffle Hopf algebras, thanks to a universal property satisfied by Connes-Kreimer Hopf algebra. We give a straightforward characterization of the fundamental process of homogeneous coarborification, using the explicit duality between decorated Connes-Kreimer and Grossman-Larson Hopf algebras. Finally, we introduce a new Hopf algebra that systematically underlies the calculations for the normalization of local dynamical systems.

RÉSUMÉ. – Nous donnons une description complète et naturelle du formalisme d'arborification/coarborification d'Ecalle en termes d'algèbres de Hopf. L'arborification apparaît alors comme une factorisation de caractères, impliquant les algèbres shuffle ou quasishuffle, en vertu d'une propriété universelle satisfaite par l'algèbre de Connes-Kreimer. Dans ce cadre, nous obtenons de façon directe le procédé fondamental de coarborification homogène, en utilisant la dualité explicite entre les algèbres de Hopf décorées de Connes-Kreimer et Grossman-Larson. Enfin, nous introduisons une nouvelle algèbre de Hopf qui est sous-jacente aux calculs de normalisation des systèmes dynamiques locaux.

1. Introduction

The local study of dynamical systems, through normalizing transformations, involves calculations in groups, or pseudogroups, of diffeomorphisms (e. g. formal, or analytic) that are tangent to identity. Other situations where these explicit calculations are required are also numerous in key questions of classification of singular geometric structures. Another source of examples is given by the so called mechanism of Birkhoff decomposition ([26]). The group $\boldsymbol{G}$ of formal tangent to Identity diffeomorphisms is the one in which most of the calculations are to be performed.

To tackle problems of this kind, Jean Ecalle has developed a powerful combinatorial environment, named *mould calculus*, that leads to formulas that are surprisingly explicit. This calculus has lately been the object of attention within the algebraic combinatorics community ([6], [7]). However, despite its striking achievements, this formalism has been

0012-9593/01/ doi:

little used in local dynamics, in pending problems that anyway seem out of reach of other approaches. One reason might be that it uses a sophisticated system of notations, in which a number of infinite sums are manipulated, in a way that calls for a number of proofs and explanations, that are to a certain extent still missing in the few existing papers using mould calculus. Beside this, some constructions introduced by Ecalle, though obviously appearing as extraordinarily efficient, might remain a bit mysterious; an example of this is the so called *homogeneous coarborification* ([11]), for which we are now able to give in the present paper a very natural algebraic presentation.

In fact, Ecalle's mould-comould formalism can be very naturally recast in a Hopf-algebraic setting, with the help of a number of Hopf algebras (shuffle, quasi-shuffle, their graded duals, etc) which are now widely used within algebraic combinatorics. In the present text, we show how this can be done, which makes it possible to give simple and quick proofs of important properties regarding mould calculus.

As is now well known ([12], [4]), the Hopf-algebraic formulation of computations on formal diffeomorphisms involves the so called Faà di Bruno Hopf algebra, which encodes the eponymous formula for higher order chain rule. In fact, Hopf algebraic tools and concepts have very recently become pervasive in dynamical systems, see e.g., [22] and the references therein. Now, *an essential point* is the following: the reformulation of a classification problem through the use of Faà di Bruno Hopf algebra (or, more simply, calculations on compositions of diffeomorphisms involving the Faà di Bruno formula), although satisfactory at the formal level will usually be inefficient, in the hard cases, for the question of analyticity of the series. *Indeed, in difficult situations involving resonances and/or small denominators, the formulas obtained through Faà di Bruno are most of the time not explicit enough to obtain satisfactory growth estimates on the coefficients*.

On the other hand, Ecalle's mould-comould expansions often lead to explicit coefficients but, when trying to control the size of these in a straightforward way, we often encounter systematic divergence, which claims for the introduction of something subtler.

So the need was for some sort of *intermediate Hopf algebra*, in which the algebraic calculations would still be tractable, and leading to explicit formulas from which key estimates can be obtained, to eventually get e.g., the analyticity properties we could expect. This is exactly what arborification/coarborification does. Once the original definitions of Ecalle are translated into a Hopf-algebraic setting, with the use of Connes-Kreimer Hopf algebra CK and its graded dual, it is possible to recognize that the arborification transform is nothing else that a property of factorization of characters between Hopf algebras (we perform this at the same time for the shuffle and quasishuffle cases), using the fact that CK is an initial object for Hochschild cohomology for a particular category of cogebras ([8], [13], [14]).

Thus, the universality of the arborification mechanism is directly and naturally connected with a universal property satisfied by Connes-Kreimer Hopf algebra, whose importance is by now widely acknowledged (see e. g. [12]).

The paper is organized as follows. In the next section, we recall a few basic facts concerning normalization in local dynamics, focusing on two basic situations for which it is possible to introduce all the relevant objects in a simple, yet non trivial, context. The following section is devoted to an algebraic study of the group of tangent to identity formal diffeomorphisms, introducing at this stage the Faà di Bruno Hopf algebra $\mathscr{H}_{\mathrm{FdB}}$. This section doesn't contain new results, yet we have chosen a presentation stressing the role of substitution automorphisms, and adopting a systematic way of looking at normalizing equations as equations on characters of Hopf algebras which are by now classical objects (basic terminology and facts on graded Hopf algebras are included).

Then we are ready to interpret moulds, at least the ones with symmetry properties that are met in practice, as characters or infinitesimal characters on some classical Hopf algebras, namely symmetral (resp. symmetrel) moulds as characters of the shuffle (resp. quasishuffle) Hopf algebra. This is the object of Section 4, where the basic notions regarding moulds, comoulds and their "contractions" are given.

In Section 5 the key dual mechanisms of arborification and coarborifications are introduced, and described through the introduction of CK and its graded dual, known to be isomorphic to the Grossman-Larson Hopf algebra

In fact, we show that the natural isomorphism between these two Hopf algebras leads directly, in the contexts of comoulds, to the process of *homogeneous coarborification*, which was put forward by Ecalle with very little explanation. A cautious handling of the symmetry factors of the trees is crucial, here.

In Section 6 we describe the Hopf algebra CK^+ which is ultimately used in practical calculations of normalizing transformations, for questions of classification of dynamical systems, involving resonances and small denominators. This solves at the same time an algebraic problem and a essential analytic one, regarding the growth estimates of the coefficients of the diffeomorphisms. The point of view which is enhanced in the present paper can be summed up in the following considerations:

- The systematic use of substitution automorphisms, which constitute an alternative – a very profitable one, because it is more *flexible* — to changes of variables, naturally entail a Hopf-algebraic presentation
- Calculations in the Faà di Bruno Hopf algebra are a direct mirror of the traditional approach through normalizing transformations, yet they don't yield results which are explicit enough to tackle difficult cases
- There is a *hierarchy* Sh/Qsh, CK, CK^+ of Hopf algebras, the first ones adapted to the simple formal classification results, the second one necessary for controlling the regularity of the formal constructions, under a strong non resonance condition, and the last one to take care of objects satisfying a weak nonresonance condition

The main results of the text are thus the ones which concern the Hopf algebra CK^+, which is the fundamental one to be used by the practitioner, in difficult problems involving small denominators.

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