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VERSIONS OF INJECTIVITY AND EXTENSION THEOREMS

BY YOSHINORI GONGYO AND SHIN-ICHI MATSUMURA

ABSTRACT. – We give an analytic version of the injectivity theorem by using multiplier ideal sheaves of singular hermitian metrics, and prove extension theorems for the log canonical bundle of dlt pairs. Moreover we obtain partial results related to the abundance conjecture in birational geometry and the semi-ampleness conjecture for hyperKähler manifolds.

RÉSUMÉ. – Nous donnons une version analytique du théorème d'injectivité en utilisant les idéaux multiplicateurs, et démontrons des théorèmes d'extension pour le faisceau adjoint d'une paire dlt. De plus nous obtenons des résultats de semi-amplitude liés à la conjecture d'abondance en géométrie birationnelle et la conjecture de semi-amplitude pour les variétés hyperkählériennes.

1. Introduction

The following conjecture, the so-called abundance conjecture, is one of the most important problems in the classification theory of algebraic varieties. In this paper, we give an analytic version of the injectivity theorem, and study the extension problem for (holomorphic) sections of the pluri-log canonical bundle and its applications to the abundance conjecture.

CONJECTURE 1.1 (Generalized abundance conjecture). – *Let X be a normal projective variety and Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is a klt pair. Then $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. In particular, if $K_X + \Delta$ is nef, then it is semi-ample.* (*See* [38] *for the definition of $\kappa(\cdot)$ and $\kappa_\sigma(\cdot)$.*)

Throughout this paper, we work over $\mathbb{C}$, the complex number field, and freely use the standard notation in [4], [7], [27], and [32]. Further we interchangeably use the words "Cartier divisors," "line bundles," and "invertible sheaves".

Toward the abundance conjecture, we need to solve the non-vanishing conjecture and the extension conjecture (for example, see [8], [12, Introduction], and [20, Section 5]). One of

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the purposes of this paper is to study the following extension conjecture formulated in [8, Conjecture 1.3]:

CONJECTURE 1.2 (Extension conjecture for dlt pairs). – *Let X be a normal projective variety and $S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions:*

- *$(X, S + B)$ is a dlt pair.*
- *$\lfloor S + B \rfloor = S$.*
- *$K_X + S + B$ is nef.*
- *$K_X + S + B$ is $\mathbb{Q}$-linearly equivalent to an effective divisor D such that*

$$S \subseteq \operatorname{Supp} D \subseteq \operatorname{Supp}(S + B).$$

Then the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

When S is a normal irreducible variety (that is, $(X, S+B)$ is a plt pair), Demailly-Hacon-Păun have already proved the above conjecture in [8] by using techniques based on a version of the Ohsawa-Takegoshi L^2 extension theorem. However, the extension theorem for plt pairs is not enough for an inductive proof of the abundance conjecture.

In this paper, we study the extension conjecture for *dlt pairs* by giving an analytic version of the injectivity theorem instead of the Ohsawa-Takegoshi extension theorem. Thanks to our injectivity theorem, we can obtain extension theorems for not only plt pairs but also dlt pairs. This is one of the advantages of our approach. The following result is our injectivity theorem.

THEOREM 1.3 (Analytic version of the injectivity theorem: Theorem 3.1)
Let (F, h_F) and (L, h_L) be (possibly) singular hermitian line bundles with semi-positive curvature on a compact Kähler manifold X. Assume that there exists an effective $\mathbb{R}$-divisor Δ with

$$h_F = h_L^a \cdot h_\Delta,$$

where a is a positive real number and h_Δ is the singular (hermitian) metric defined by the effective divisor Δ.

Then, for a non-zero (holomorphic) section s of L satisfying $\sup_X |s|_{h_L} < \infty$, the multiplication map induced by s

$$H^q(X, K_X \otimes F \otimes \mathcal{J}(h_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes L \otimes \mathcal{J}(h_F h_L))$$

is (well-defined and) injective for every q. Here $\mathcal{J}(h)$ denotes the multiplier ideal sheaf associated to a singular (hermitian) metric h.

In the last decades, the injectivity theorem has been studied by several authors, for example, Tankeev [40], Kollár [29], Enoki [10], Ohsawa [39], Esnault-Viehweg [11], Fujino [18], [14], [16], [17], and Ambro [1], [2]. See [19] and [35] for recent developments. Theorem 1.3 can be seen as a generalization of [10], [16], [29], [34], [37], and [40]. In [37], the second author established an injectivity theorem with multiplier ideal sheaves of singular (hermitian) metrics with arbitrary singularities, which corresponds to the case $\Delta = 0$ of Theorem 1.3.

By applying the above injectivity theorem to the extension problem, we obtain the following extension theorem. Even if $K_X + \Delta$ is semi-positive (namely, it admits a smooth hermitian metric with semi-positive curvature), it seems to be rather difficult to obtain the extension theorem for dlt pairs by the Ohsawa-Takegoshi extension theorem, at least in its present forms. This is because there exists a counterexample to the Ohsawa-Takegoshi extension theorem for dlt pairs (see [39, page 576]). For this reason, we need our injectivity theorem.

THEOREM 1.4 (Extension theorem: Theorem 4.1). – *Let X be a compact Kähler manifold and $S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions:*

- *$S + B$ is a simple normal crossing divisor with $0 \leq S + B \leq 1$ and $\lfloor S + B \rfloor = S$.*
- *$K_X + S + B$ is $\mathbb{Q}$-linearly equivalent to an effective divisor D with $S \subseteq \operatorname{Supp} D$.*
- *$K_X + S + B$ admits a singular (hermitian) metric h with semi-positive curvature.*

Then, for an integer $m \geq 2$ with $m(K_X + S + B)$ Cartier and a section

$$u \in H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

that comes from $H^0(S, \mathcal{O}_S(m(K_X + S + B)) \otimes \mathcal{J}(h^{m-1}h_B))$, the section u can be extended to a section in $H^0(X, \mathcal{O}_X(m(K_X + S + B)))$.

By this theorem, we can solve the extension problem for dlt pairs if there exists a singular (hermitian) metric with mild singularities on S (see Corollary 4.2 and Corollary 4.4). When we consider the extension problem, we first construct a (possibly singular) hermitian metric with "good" properties by taking the limit of a family of suitable metrics. In the second step, we extend sections by using the metric constructed in the first step. Currently we do not know the first step to construct a suitable metric. However we can solve the second step by Theorem 1.4 and its corollaries.

Moreover, assuming the non-vanishing conjecture, we can prove the abundance conjecture if $K_X + \Delta$ admits a singular (hermitian) metric h whose curvature is semi-positive and Lelong number is identically zero. This assumption is stronger than the assumption that $K_X + \Delta$ is nef, but weaker than the assumption that $K_X + \Delta$ is semi-positive. To investigate the Lelong number is much easier than to check the regularity (smoothness) of the metric constructed by taking the limit. Therefore it is worth formulating our extension theorem for a singular (hermitian) metric h whose Lelong number is identically zero (see Corollary 4.2 and Corollary 4.4).

As compared with Conjecture 1.2, one of our advantages is to remove the condition $\operatorname{Supp} D \subseteq \operatorname{Supp}(S + B)$ in Conjecture 1.2, which is needed in [8]. (Such an extension conjecture was given in [20, Conjecture 5.8]). Thanks to removing this condition, we can apply the extension theorem more directly than [8, Section 8] and [20, Theorem 5.9], and construct a (non-klt) dlt birational model whose log canonical divisor is a pullback of the original canonical divisor up to positive multiples. Therefore we finally obtain the following theorem related to the abundance conjecture:

THEOREM 1.5 (Partial result of the abundance conjecture, cf. Theorem 5.1)
Assume that Conjecture 1.1 holds in dimension $(n - 1)$. Let X be an n-dimensional normal projective variety and Δ be an effective $\mathbb{Q}$-divisor with the following assumptions: