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HIGHER DIMENSIONAL FORMAL LOOP SPACES

BY BENJAMIN HENNION

ABSTRACT. – If M is a symplectic manifold then the space of smooth loops $\mathrm{C}^{\infty}(\mathrm{S}^1, M)$ inherits of a quasi-symplectic form. We will focus in this article on an algebraic analog of that result. In their article [14], Kapranov and Vasserot introduced and studied the formal loop space of a scheme X.

We generalize their construction to higher dimensional loops. To any scheme X—not necessarily smooth—we associate $\mathcal{L}^d(X)$, the space of loops of dimension d. We prove it has a structure of (derived) Tate scheme—i.e., its tangent is a Tate module: it is infinite dimensional but behaves nicely enough regarding duality. We also define the bubble space $\mathfrak{B}^d(X)$, a variation of the loop space. We prove that $\mathfrak{B}^d(X)$ is endowed with a natural symplectic form as soon as X has one (in the sense of [22]).

Throughout this paper, we will use the tools of $(\infty, 1)$-categories and symplectic derived algebraic geometry.

RÉSUMÉ. – L'espace des lacets lisses $\mathrm{C}^{\infty}(\mathrm{S}^1, M)$ associé à une variété symplectique M se voit doté d'une structure (quasi-)symplectique induite par celle de M. Nous traiterons dans cet article d'un analogue algébrique de cet énoncé. Dans leur article [14], Kapranov et Vasserot ont introduit l'espace des lacets formels associé à un schéma.

Nous généralisons leur construction à des lacets de dimension supérieure. Nous associons à tout schéma X — pas forcément lisse — l'espace $\mathcal{L}^d(X)$ de ses lacets formels de dimension d. Nous démontrerons que ce dernier admet une structure de schéma (dérivé) de Tate : son espace tangent est de Tate : de dimension infinie mais suffisamment structuré pour se soumettre à la dualité. Nous définirons également l'espace $\mathfrak{B}^d(X)$ des bulles de X, une variante de l'espace des lacets, et nous montrerons que le cas échéant, il hérite de la structure symplectique de X.

Introduction

Considering a differential manifold M, one can build the space of smooth loops $\mathrm{L}(M)$ in M. It is a central object of string theory. Moreover, if M is symplectic then so is $\mathrm{L}(M)$—more precisely quasi-symplectic since it is not of finite dimension—see for instance [20]. We will be interested here in an algebraic analog of that result.

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The first question is then the following: what is an algebraic analog of the space of smooth loops? An answer appeared in 1994 in Carlos Contou-Carrère's work (see [6]). He studies there $\mathbb{G}_m(\mathbb{C}(\!(t)\!))$, some sort of holomorphic functions in the multiplicative group scheme, and defines the famous Contou-Carrère symbol. This is the first occurrence of a *formal loop space* known to the author. This idea was then generalized to algebraic groups as the affine Grassmannian $\mathfrak{Gr}_G = G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$ showed up and got involved in the geometric Langlands program. In their paper [14], Mikhail Kapranov and Éric Vasserot introduced and studied the formal loop space of a smooth scheme X. It is an ind-scheme $\mathcal{L}(X)$ which we can think of as the space of maps $\operatorname{Spec}\mathbb{C}(\!(t)\!) \to X$. This construction strongly inspired the one presented in this article.

There are at least two ways to build higher dimensional formal loops. The most studied one consists in using higher dimensional local fields $k(\!(t_1)\!)\dots(\!(t_d)\!)$ and is linked to Beilinson's adèles. There is also a generalization of Contou-Carrère symbol in higher dimensions using those higher dimensional local fields—see [21] and [5]. If we had adopted this angle, we would have considered maps from some torus [(1)] $\operatorname{Spec}(k(\!(t_1)\!)\dots(\!(t_d)\!))$ to X.

The approach we will follow in this work is different. We generalize here the definition of Kapranov and Vasserot to higher dimensional loops in the following way. For X a scheme of finite presentation, not necessarily smooth, we define $\mathcal{L}^d(X)$, the space of formal loops of dimension d in X. We define $\mathcal{L}^d_V(X)$ the space of maps from the formal neighborhood of 0 in $\mathbb{A}^d$ to X. This is a higher dimensional version of the space of germs of arcs as studied by Jan Denef and François Loeser in [7]. Let also $\mathcal{L}^d_U(X)$ denote the space of maps from a *punctured* formal neighborhood of 0 in $\mathbb{A}^d$ to X. The formal loop space $\mathcal{L}^d(X)$ is the formal completion of $\mathcal{L}^d_V(X)$ in $\mathcal{L}^d_U(X)$. Understanding those three items is the main goal of this work. The problem is mainly to give a meaningful definition of the punctured formal neighborhood of dimension d. We can describe what its cohomology should be:

$$\mathrm{H}^n(\hat{\mathbb{A}}^d \smallsetminus \{0\}) = \begin{cases} k[\![X_1,\dots,X_d]\!] & \text{if } n=0 \\ (X_1\dots X_d)^{-1}k[X_1^{-1},\dots,X_d^{-1}] & \text{if } n=d-1 \\ 0 & \text{otherwise} \end{cases}$$

but defining this punctured formal neighborhood with all its structure is actually not an easy task. Nevertheless, we can describe what maps out of it are, hence the definition of $\mathcal{L}^d_U(X)$ and the formal loop space. This geometric object is of infinite dimension, and part of this study is aimed at identifying some structure. Here comes the first result in that direction.

Theorem 1 (See 3.3.4). – *The formal loop space of dimension d in a scheme X is represented by a derived ind-pro-scheme. Moreover, the functor $X \mapsto \mathcal{L}^d(X)$ satisfies the étale descent condition.*

We use here methods from derived algebraic geometry as developed by Bertrand Toën and Gabriele Vezzosi in [25]. The author would like to emphasize here that the derived structure is necessary since, when X is a scheme, the underlying schemes of $\mathcal{L}^d(X)$, $\mathcal{L}^d_U(X)$ and $\mathcal{L}^d_V(X)$ are isomorphic as soon as $d \geq 2$. Let us also note that derived algebraic geometry allowed

(1) The variable $t_1,\dots,t_d$ are actually ordered. The author likes to think of $\operatorname{Spec}(k(\!(t_1)\!)\dots(\!(t_d)\!))$ as a formal torus equipped with a flag representing this order.

us to define $\mathcal{L}^d(X)$ for more general X's, namely any derived stack. In this case, the formal loop space $\mathcal{L}^d(X)$ is no longer a derived ind-pro-scheme but an ind-pro-stack.

The case $d = 1$ and X is a smooth scheme gives a derived enhancement of Kapranov and Vasserot's definition. This derived enhancement is conjectured to be trivial when X is a smooth affine scheme in [8, 9.2.10]. Gaitsgory and Rozenblyum also prove in *loc. cit.* their conjecture holds when X is an algebraic group.

The proof of Theorem 1 is based on an important lemma. We identify a full sub-category $\mathcal{C}$ of the category of ind-pro-stacks such that the realization functor $\mathcal{C} \to \mathbf{dSt}_k$ is fully faithful. We then prove that whenever X is a derived affine scheme, the stack $\mathcal{L}^d(X)$ is in the essential image of $\mathcal{C}$ and is thus endowed with an *essentially unique* ind-pro-structure satisfying some properties. The generalization to any X is made using a descent argument. Note that for general X's, the ind-pro-structure is not known to satisfy nice properties one could want to have, for instance on the transition maps of the diagrams.

We then focus on the following problem: can we build a symplectic form on $\mathcal{L}^d(X)$ when X is symplectic? Again, this question requires the tools of derived algebraic geometry and *shifted symplectic structures* as in [22]. A key feature of derived algebraic geometry is the cotangent complex $\mathbb{L}_X$ of any geometric object X. A (n-shifted) symplectic structure on X is a closed 2-form $\mathcal{O}_X[-n] \to \mathbb{L}_X \wedge \mathbb{L}_X$ which is non degenerate—i.e., induces an equivalence

$$\mathbb{T}_X \to \mathbb{L}_X[n].$$

Because $\mathcal{L}^d(X)$ is not finite, linking its cotangent to its dual—through an alleged symplectic form—requires to identify once more some structure. We already know that it is an ind-pro-scheme but the proper context seems to be what we call Tate stacks.

Before saying what a Tate stack is, let us talk about Tate modules. They define a convenient context for infinite dimensional vector spaces. They where studied by Lefschetz, Beilinson and Drinfeld, among others, and more recently by Bräunling, Gröchenig and Wolfson [4]. We will use here the notion of Tate objects in the context of stable $(\infty, 1)$-categories as developed in [11]. If $\mathcal{C}$ is a stable $(\infty, 1)$-category—playing the role of the category of finite dimensional vector spaces, the category $\mathbf{Tate}(\mathcal{C})$ is the full subcategory of the $(\infty, 1)$-category of pro-ind-objects $\mathbf{Pro}\,\mathbf{Ind}(\mathcal{C})$ in $\mathcal{C}$ containing both $\mathbf{Ind}(\mathcal{C})$ and $\mathbf{Pro}(\mathcal{C})$ and stable by extensions and retracts.

We will define the derived category of Tate modules on a scheme—and more generally on a derived ind-pro-stack. An Artin ind-pro-stack X—meaning an ind-pro-object in derived Artin stacks—is then gifted with a cotangent complex $\mathbb{L}_X$. This cotangent complex inherits a natural structure of pro-ind-module on X. This allows us to define a Tate stack as an Artin ind-pro-stack whose cotangent complex is a Tate module. The formal loop space $\mathcal{L}^d(X)$ is then a Tate stack as soon as X is a finitely presented derived affine scheme. For a more general X, what precedes makes $\mathcal{L}^d(X)$ some kind of *locally* Tate stack. This structure suffices to define a determinantal anomaly

$$\left[\mathrm{Det}_{\mathcal{L}^d(X)}\right] \in \mathrm{H}^2\left(\mathcal{L}^d(X), \mathcal{O}^{\times}_{\mathcal{L}^d(X)}\right)$$

for any quasi-compact quasi-separated (derived) scheme X—this construction also works for slightly more general X's, namely Deligne-Mumford stacks with algebraizable diagonal,