ISSN 0012-9593

quatrième série - tome 50 fascicule 4 juillet-août 2017

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 50, 2017, p. 995 à 1016

KNOTTED STRUCTURES IN HIGH-ENERGY BELTRAMI FIELDS ON THE TORUS AND THE SPHERE

BY Alberto ENCISO, Daniel PERALTA-SALAS AND Francisco TORRES de LIZAUR

ABSTRACT. – Let $\mathcal{S}$ be a finite union of (pairwise disjoint but possibly knotted and linked) closed curves and tubes in the round sphere $\mathbb{S}^3$ or in the flat torus $\mathbb{T}^3$. In the case of the torus, $\mathcal{S}$ is further assumed to be contained in a contractible subset of $\mathbb{T}^3$. In this paper we show that for any sufficiently large odd integer λ there exists a Beltrami field on $\mathbb{S}^3$ or $\mathbb{T}^3$ satisfying $\operatorname{curl} u = \lambda u$ and with a collection of vortex lines and vortex tubes given by $\mathcal{S}$, up to an ambient diffeomorphism.

RÉSUMÉ. – Soit $\mathcal{S}$ une collection finie de courbes et de tubes fermés, disjoints deux à deux mais pouvant être noués et entrelacés, dans la sphère ronde $\mathbb{S}^3$ ou dans le tore plat $\mathbb{T}^3$. Dans le cas du tore, on suppose davantage que $\mathcal{S}$ est contenu dans un sous-ensemble contractile de $\mathbb{T}^3$. Dans cet article on montre que, pour tout entier impair λ suffisamment grand, il existe un champ de Beltrami dans $\mathbb{S}^3$ ou $\mathbb{T}^3$ satisfaisant $\operatorname{curl} u = \lambda u$ et qui a une collection de lignes et tubes de vorticité donnés par $\mathcal{S}$, modulo un difféomorphisme ambiant.

1. Introduction

An incompressible fluid flow in $\mathbb{R}^3$ is described by its velocity field $u(x,t)$, which is a time-dependent vector field satisfying the Euler equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla P\,, \qquad \operatorname{div} u = 0$$

for some pressure function $P(x,t)$. When the velocity field does not depend on time, the fluid is said to be *stationary*. This paper concerns stationary solutions of the Euler equations, which describe equilibrium configurations of the fluid.

A central topic in topological fluid mechanics, which can be traced back to Lord Kelvin in the 19th century [20], concerns the existence of knotted stream and vortex structures in stationary fluid flows. The most relevant of these structures are the stream lines, vortex lines and vortex tubes of the fluid. We recall that a *stream line* and a *vortex line* are simply a trajectory (or integral curve) of the velocity field u and the vorticity $\omega := \operatorname{curl} u$, respectively, while a *vortex tube* is the interior domain bounded by an invariant torus of the vorticity.

 doi:10.24033/asens.2337

The existence of topologically complicated stream and vortex lines is a central topic in the Lagrangian theory of turbulence and in magnetohydrodynamics, and has been studied extensively in the last decades (see e.g., [13, 15] for recent accounts of the subject).

Our understanding of the set of stationary states of the Euler equations in three dimensions is much more limited than in the two-dimensional situation [4, 17]. This is due to the fact that, in two dimensions, the vorticity is a scalar quantity, whereas in the three dimensional case it is a vector field, which can exhibit a much richer behavior. In particular, the existence of stationary solutions in $\mathbb{R}^3$ having stream lines, vortex lines and vortex tubes that are knotted and linked in arbitrarily complicated ways has been established only very recently [7, 8, 9]. Following a suggestion of Arnold [2, 1] related to his celebrated structure theorem, to prove these results one does not consider just any kind of solutions to the stationary Euler equations, but a very particular class that are called Beltrami fields. A *Beltrami field* in $\mathbb{R}^3$ is a vector field satisfying the equation

$$\operatorname{curl} u = \lambda u \tag{1.1}$$

for some nonzero constant λ. Notice that stream lines and vortex lines coincide in the case of a Beltrami field, and that a Beltrami field is automatically smooth (even real analytic) by the elliptic regularity theory.

The stationary solutions in $\mathbb{R}^3$ that one can construct using the techniques in [7, 8] fall off at infinity as $1/|x|$, this decay being sharp for Beltrami fields but not fast enough for the velocity to be in the energy space $L^2(\mathbb{R}^3)$. In fact, the incompressibility condition ensures that there are no Beltrami fields in $\mathbb{R}^3$ with finite energy even if the proportionality factor λ is allowed to be nonconstant, as has been recently shown in [18, 3].

On the contrary, Beltrami fields in a closed Riemannian 3-manifold M (or a bounded domain of $\mathbb{R}^3$) are stationary solutions to the Euler equations that do have finite energy. If $\mathcal{S}$ is a union of (possibly knotted and linked) closed curves and embedded tori in the 3-sphere, in this setting one can use contact topology to show [10] that there is a Riemannian metric g on the sphere with an associated Beltrami field u having a collection of vortex lines and vortex tubes given precisely by $\mathcal{S}$. The main ideas of the proof are that the Reeb field of a contact form is in fact a Beltrami field in some adapted metric and that one can indeed construct contact forms on the sphere whose Reeb fields have the collection of periodic trajectories and invariant tori given by $\mathcal{S}$. Notice that, as it is a Reeb vector field, a Beltrami field obtained in this fashion does not vanish. Conversely, any nonvanishing Beltrami field on the sphere is the Reeb vector field of some contact form.

Our goal in this paper is to establish the existence of knotted and linked vortex structures in Beltrami fields on compact manifolds with a *fixed* Riemannian metric. Specifically, we will consider Beltrami fields in the flat 3-torus $\mathbb{T}^3$ and in the unit 3-sphere $\mathbb{S}^3$; in fact, the former is the most fundamental space considered in the fluid mechanics literature other than $\mathbb{R}^3$ and the latter is perhaps the simplest example of a closed Riemannian 3-manifold from a geometric point of view.

It is worth emphasizing that, for a fixed Riemannian structure, the problem is much more rigid than when one can freely choose a metric adapted to the geometry of the set of lines and tubes that one aims to recover from the trajectories of a Beltrami field. An obvious reason is that, analytically, Beltrami fields in a closed Riemannian manifold arise as eigenfields of the curl operator, which defines a self-adjoint operator with discrete spectrum and a dense domain in the space of divergence-free L^2 fields. In the context of spectral theory, the proportionality constant λ, or rather its absolute value, can be thought of as the *energy* of the Beltrami field, although of course it is in no way related to the L^2 norm of the latter.

Our main theorem asserts that there are "many" Beltrami fields u in the sphere and in the torus with vortex lines and vortex tubes of any link type. Furthermore, these structures are *structurally stable* in the sense that any vector field on the torus or the sphere which is sufficiently close to u in the C^4 norm and which preserves some smooth volume measure will also have this collection of periodic trajectories and invariant tori, up to a diffeomorphism. To state this result precisely, let us call a *tube* the closure of a domain (in $\mathbb{S}^3$ or $\mathbb{T}^3$) whose boundary is an embedded torus. Throughout, diffeomorphisms are of class C^∞, curves are all assumed to be non-self-intersecting, and we will agree to say that an integer is large when it is large in absolute value.

THEOREM 1.1. – *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{S}^3$ or $\mathbb{T}^3$. In the case of the torus, we also assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation* $\operatorname{curl} u = \lambda u$ *and a diffeomorphism Φ of $\mathbb{S}^3$ or $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

An important observation is that the proof of this theorem yields a reasonably complete understanding of the behavior of the diffeomorphism Φ, which is, in particular, connected with the identity. Oversimplifying a little, the effect of Φ is to uniformly rescale a contractible subset of the manifold that contains $\mathcal{S}$ to have a diameter of order $1/|\lambda|$. In particular, the control that we have over the diffeomorphism Φ allows us to prove an analog of this result for quotients of the sphere by finite groups of isometries (lens spaces). Notice that $\Phi(\mathcal{S})$ is not guaranteed to contain all vortex lines and vortex tubes of the Beltrami field. It is also worth mentioning that, if $\mathcal{S}$ only consists of curves, the condition that the perturbation of the Beltrami field be volume-preserving is not necessary for the structural stability of $\Phi(\mathcal{S})$, and the smallness in C^4 can be replaced by a C^1 condition.

In $\mathbb{S}^3$ and $\mathbb{T}^3$, Theorem 1.1 proves a conjecture of Arnold [2] asserting that there should be Beltrami fields having stream lines with complicated topology. Furthermore, it should be noticed that the helicity of the vorticity, that is, the quantity [14]

$$\mathcal{H}(\operatorname{curl} u) := \int_M u \cdot \operatorname{curl} u \,,$$

is proportional to its eigenvalue λ, so the Beltrami fields constructed in the main theorem have very large helicity. More precisely, the quantity

$$\frac{\mathcal{H}(\operatorname{curl} u)}{\|u\|_{L^2}^2} \,,$$