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KISIN MODULES WITH DESCENT DATA AND PARAHORIC LOCAL MODELS

BY ANA CARAIANI AND BRANDON LEVIN

ABSTRACT. – We construct a moduli space $Y^{\mu,\tau}$ of Kisin modules with tame descent datum τ and with p-adic Hodge type $\leq \mu$, for some finite extension $K/\mathbb{Q}_p$. We show that this space is smoothly equivalent to the local model for $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n$, cocharacter $\{\mu\}$, and parahoric level structure. We use this to construct the analog of Kottwitz-Rapoport strata on the special fiber $Y^{\mu,\tau}$ indexed by the μ-admissible set. We also relate $Y^{\mu,\tau}$ to potentially crystalline Galois deformation rings.

RÉSUMÉ. – Nous construisons un espace de modules $Y^{\mu,\tau}$ de modules de Kisin avec donnée de descente modérée τ et type de Hodge p-adique μ, pour une extension finie $K/\mathbb{Q}_p$. Nous démontrons une équivalence lisse entre $Y^{\mu,\tau}$ et le modèle local pour la restriction de scalaires $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n$, cocaractère $\{\mu\}$ et structure de niveau parahorique. Cette équivalence est ensuite utilisée pour construire l'analogue de la stratification de Kottwitz-Rapoport sur la fibre spéciale de $Y^{\mu,\tau}$, paramétrée par l'ensemble des éléments μ-admissibles. Nous décrivons aussi la relation entre $Y^{\mu,\tau}$ et l'espace de déformations galoisiennes potentiellement cristallines.

1. Introduction

Let $K/\mathbb{Q}_p$ be a finite extension. Kisin [22] showed that the category of finite flat commutative group schemes over $\mathcal{O}_K$ killed by a power of p is equivalent to the category of *Breuil-Kisin modules* of height ≤ 1. While the former do not naturally live in families, one can work with Breuil-Kisin modules with coefficients and study their moduli. The landmark paper [25] uses moduli of Breuil-Kisin modules to construct resolutions of flat deformation rings with stunning consequences for modularity lifting theorems and applications to the Fontaine-Mazur conjecture. The main result of [25] is a modularity lifting theorem in the potentially Barsotti-Tate case. One of the key points is a rather surprising connection to the theory of *local models of Shimura varieties*. Kisin showed that the singularities of the moduli space of Breuil-Kisin modules of rank n (with fixed p-adic Hodge type) could be related to the singularities of local models for the group $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n$ (with maximal parahoric level) which had been studied by [32].

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Kisin's result is globalized in [33], where Pappas and Rapoport construct a global (formal) moduli stack X^μ of Kisin modules with p-adic Hodge type $\mu \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$. They link the space X^μ via smooth maps with a (generalized) local model $M(\mu)$. When μ is non-minuscule, $M(\mu)$ is not related to any Shimura variety but is nevertheless known to have nice geometric properties by work of Pappas-Zhu [35] and of the second author [28]. $M(\mu)$ is constructed inside a mixed characteristic version of the Beilinson-Drinfeld affine Grassmannian. As a result, the nice geometric properties of $M(\mu)$ transfer to the global moduli stack X^μ.

While the connection between moduli of Breuil-Kisin modules and local models suffices for proving modularity lifting theorems in the potentially Barsotti-Tate case, it doesn't seem capture some of the more subtle aspects of the geometry of local deformation rings. These more subtle aspects are connected to the (geometric) Breuil-Mezard conjecture [6, 14], to the weight part in Serre's conjecture [8, 18] and to questions about integral structures in completed cohomology [5, 16]. Therefore, there is considerable interest in generalizing the results of Kisin and Pappas-Rapoport. This paper extends the relationship with local models to the case of Breuil-Kisin modules equipped with *tame descent data*.

We explain the connection to integral structures in completed cohomology. One of the few situations where we have explicit presentations of local deformation rings is the case of tamely Barsotti-Tate deformations rings for GL_2. Set $G_K := \mathrm{Gal}(\bar{K}/K)$ and let $I_K \subset G_K$ be the inertia subgroup. When $K/\mathbb{Q}_p$ is unramified and $\tau : I_K \to \mathrm{GL}_2(\Lambda)$ is a (generic) tame *inertial type*, then [5, 7, 16] explicitly describe the potentially Barsotti-Tate deformation ring $R_{\overline{\rho}}^{\mathrm{BT},\tau}$ for any $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$. These computations provided evidence for the Breuil-Mézard conjecture and led Breuil to several important conjectures [5]. Perhaps the most striking is the precise conjecture about which lattices inside the smooth $GL_2(\mathcal{O}_K)$-representation $\sigma(\tau)$ (determined by τ via inertial local Langlands) can occur globally, in completed cohomology. Breuil's conjectures were proved by Emerton-Gee-Savitt [16] using the explicit presentations of tamely Barsotti-Tate deformation rings.

In more general situations ($K/\mathbb{Q}_p$ ramified or $\overline{\rho}$ non-generic), one cannot hope for such an explicit presentation. In this paper, we construct for arbitrary $K/\mathbb{Q}_p$ and GL_n, resolutions of tamely Barsotti-Tate deformation rings whose geometry is related to that of local models for $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n$ with parahoric level structure. These resolutions are related to the moduli of Breuil-Kisin modules with descent data. The level structure is determined by the tame inertial type τ. For example, if τ consists of distinct characters, then the local model will have Iwahori level structure, whereas the local models of [25, 33], which have trivial descent data, always have maximal parahoric level.

Our perspective in this paper is largely global, in the spirit of [33]. Motivated by the moduli stack of finite flat representations of G_K constructed by [15], we study moduli stacks $Y^{\mu,\tau}$ of Kisin modules with tame descent data and p-adic Hodge type $\mu \in (\mathbb{Z}^n)^{\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$. We can consider a moduli stack of Kisin modules as above, but in addition equipped with an eigenbasis compatible with the descent datum; we call this space $\widetilde{Y}^{\mu,\tau}$.

THEOREM 1.1. – *There exists a moduli stack $Y^{\mu,\tau}$ of Kisin modules with tame descent data and p-adic Hodge type μ, which fits into the diagram*

$$\begin{tikzcd} & \widetilde{Y}^{\mu,\tau} \arrow[dl, "\pi^{\mu}"'] \arrow[dr, "\Psi^{\mu}"] & \\ Y^{\mu,\tau} & & M(\mu), \end{tikzcd}$$

where $M(\mu)$ is the Pappas-Zhu local model [35, 28] *for* $(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n, \mu)$ *at parahoric level* (*determined by τ*) *and both π^{μ} and Ψ^{μ} are smooth maps.*

REMARK 1.2. – The key step in the construction of the local model diagram is encoded in diagram 3.1. We decompose a Kisin module $(\mathfrak{M}, \phi)$ according to the descent datum and then study the interactions between the images of ϕ on different isotopic pieces. This is reminiscent of the classical definition of local models which involves lattice chains.

REMARK 1.3. – The main idea behind constructing the local model diagram in Theorem 1.1 comes by observing that there is a correspondence between having descent datum from the ramified extension L down to K and having a parahoric level structure defined over K. This relationship also appears in the theory of vector bundles over a curve, where a vector bundle with descent datum over a ramified cover of a curve corresponds to a parahoric vector bundle over the curve.

For example, the paper [29] studies the case of vector bundles over smooth projective curves X over $\mathbb{C}$. Assume X has genus ≥ 2. There exists a simply connected covering of X ramified at a finite set of points (the points and their ramification indices can be prescribed in advance) and this covering can be identified with the upper half space $\mathbb{H}$. We can identify $X = \mathbb{H}/\pi$, where π is a group of automorphisms of $\mathbb{H}$ which *does not* act freely on $\mathbb{H}$. Giving a vector bundle of rank n on $\mathbb{H}$ with descent datum to X amounts to giving the trivial rank n bundle on $\mathbb{H}$ together with a homomorphism $\pi \to GL_n(\mathbb{C})$ which induces an action of π on the trivial bundle. The invariant direct image under the projection to X gives a vector bundle on X together with a so-called *parabolic structure*. The parabolic structure consists of assigning a flag and a set of weights to the fiber at every ramification point. This construction gives an equivalence between the category of vector bundles on $\mathbb{H}$ with descent datum to X and the category of vector bundles on X with parabolic structure and with rational weights.

We also note that the more recent paper [2] extends the results of [29] to the case where the structure group is a semisimple simply-connected algebraic group over $\mathbb{C}$ (rather than GL_n).

In joint work in preparation with Emerton, Gee and Savitt [10], the first author constructs a moduli stack of two-dimensional, tamely potentially Barsotti-Tate G_K-representations and relates its geometry to the weight part of Serre's conjecture. In this case, the stack $Y^{\mu,\tau}$ will be a relatively explicit, partial resolution of the moduli stack of G_K-representations. The nice geometric properties that $Y^{\mu,\tau}$ inherits from the local model diagram turn out to be key for understanding the geometry of the latter moduli stack. From this perspective, the present paper and the paper in preparation [10] clarify the geometry which underlies a possible generalization of Breuil's lattice conjecture in the ramified setting.