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CANONICAL SMOOTHING OF COMPACT ALEKSANDROV SURFACES VIA RICCI FLOW

BY THOMAS RICHARD

ABSTRACT. – In this paper, we show existence and uniqueness of Ricci flow whose initial condition is a compact Aleksandrov surface with curvature bounded from below. This requires a weakening of the notion of initial condition which is able to deal with a priori non-Riemannian metric spaces. As a by-product, we obtain that the Ricci flow of a surface depends smoothly on Gromov-Hausdorff perturbations of the initial condition.

RÉSUMÉ. – Dans cet article, on montre l'existence et l'unicité du flot de Ricci avec pour condition initiale une surface d'Aleksandrov compacte à courbure minorée. Cela nécessite un affaiblissement de la notion de condition initiale permettant de considérer des espaces métriques a priori non riemanniens. Comme corollaire, on montre que le flot de Ricci d'une surface compacte dépend lissement des perturbations de sa condition initiale au sens de Gromov-Hausdorff.

Introduction

Ricci flow of smooth manifolds has had strong applications to the study of smooth Riemannian manifolds. It is therefore natural to ask if Ricci flow can be helpful in the study of non-smooth geometric objects. A reasonable assumption to make on a metric space (X, d) that we want to deform by the Ricci flow is to require (X, d) to be approximated in some sense by a sequence (M_i, g_i) of smooth Riemannian manifolds. In [13] and [14], M. Simon studied a class of 3-dimensional metric spaces by this method. An important feature of such "Ricci flows of metric spaces" is that the notion of initial condition has to be weakened. In the work of M. Simon [13] and [14], and of the author [12], a weak notion of initial condition has been used, which we call "metric initial condition":

DEFINITION 0.1. – *A Ricci flow $(M, g(t))_{t\in(0,T)}$ on a compact manifold M is said to have the metric space (X, d) as metric initial condition if the Riemannian distances $d_{g(t)}$ uniformly converge as t goes to 0 (as functions $M \times M \to \mathbb{R}$) to a distance $\tilde{d}$ on M such that $(M, \tilde{d})$ is isometric to (X, d).*

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REMARK 0.2. – The compactness assumption in the definition gives that (X, d) is homeomorphic to M with its manifold topology. This follows from the fact that $\tilde{d}$ is continuous on M, which implies that the identity of M is continuous as a map from M with its usual topology to M with the topology defined by $\tilde{d}$, compactness of M then gives that the identity is a homeomorphism.

The existence of such flows for some classes of metric spaces (X, d) has been proved in [13, 14] and [12]. An interesting class of spaces for which existence holds is the class of compact Aleksandrov surfaces whose curvature is bounded from below.

DEFINITION 0.3. – *A compact Aleksandrov surface whose curvature is bounded from below is a geodesic metric space (X, d) which is at the same time a compact topological surface (without boundary) and a metric space with curvature bounded from below in the sense of Aleksandrov.*

REMARK 0.4. – A geodesic metric space has curvature greater than $k \in \mathbb{R}$ in the sense of Aleksandrov if its geodesic triangles are bigger than the geodesic triangles in the complete simply connected surface $\mathbb{S}^2_k$ with curvature k.

To be more precise, a geodesic metric space (X, d) has curvature greater than k in the sense of Aleksandrov if and only if the following condition is satisfied:

Let a, b, c be any three points in (X, d), and m be any point on a shortest path from b to c. Let $\tilde{a}, \tilde{b}, \tilde{c}$ be points in $\mathbb{S}^2_k$ such that $d_k(\tilde{a}, \tilde{b}) = d(a, b)$, $d_k(\tilde{a}, \tilde{c}) = d(a, c)$ and $d_k(\tilde{b}, \tilde{c}) = d(b, c)$ where d_k is the usual distance in $\mathbb{S}^2_k$, and $\tilde{m}$ be a point on a shortest path from $\tilde{b}$ to $\tilde{c}$ such that $d_k(\tilde{b}, \tilde{m}) = d(b, m)$. Then $d(a, m) \geq d_k(\tilde{a}, \tilde{m})$.

By Toponogov's Theorem, every complete smooth surface (M, g) with Gauss curvature K_g satisfying $K_g(x) \geq k$ for every $x \in M$ is an Aleksandrov surface with curvature bounded from below by k. Another example is the boundary X of a convex set in $\mathbb{R}^n$ (resp. $\mathbb{H}^n$), endowed with its intrinsic metric d coming from the ambient metric. It can be shown (see [4], Theorem 10.2.6) that X has curvature bounded from below by 0 (resp. -1).

A metric space (X, d) will be said to have curvature bounded from below in the sense of Aleksandrov if it has curvature greater than some $k \in \mathbb{R}$ in the sense of Aleksandrov.

For more on Aleksandrov spaces, see [4], Chapters 4 and 10.

In this paper we prove uniqueness for the Ricci flow with such surfaces as metric initial condition, more precisely:

THEOREM 0.5. – *Let $(M_1, g_1(t))_{t\in(0,T]}$ and $(M_2, g_2(t))_{t\in(0,T]}$ be two smooth Ricci flows which admit a compact Aleksandrov surface (X, d) as metric initial condition. Assume furthermore that one can find $K > 0$ such that:*

$$\forall (x, t) \in M_i \times (0, T] \quad K_{g_i(t)}(x) \geq -K.$$

where $K_{g_i(t)}(x)$ is the Gauss curvature of $(M_i, g_i(t))$ at the point x.

Then there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $g_2(t) = \varphi^* g_1(t)$.

Note that the required bounds on the Ricci flow are provided by the existence proof outlined in Section 1.1.

In the next few lines, we outline the proof of Theorem 0.5. Both of the two Ricci flows $(M_i, g_i(t))$ stay in a fixed conformal class, and thus can be written $g_i(t) = w_i(x,t)h_i(x)$ for some fixed background metric h_i which can be chosen to have constant curvature. We first show that the metric initial condition prescribes the conformal class of the flow, thus we can assume that $h_1 = h_2 = h$. The proof of this fact uses deep results from the theory of singular surfaces introduced by A. D. Aleksandrov. This implies that our two Ricci flows can be seen as solutions of the following nonlinear PDE on (M, h):

$$\frac{\partial w_i}{\partial t} = \Delta_h \log(w_i) - 2K_h.$$

One then shows that w_1 and w_2 share the same L^1 initial condition as t goes to 0 and uses standard techniques to show uniqueness.

Our result can be stated in two other ways:

Proposition 0.6. – *Let M be a compact topological surface, and d be a distance on M which induces on M its manifold topology and such that (M, d) is an Aleksandrov surface with curvature bounded from below.*

Let $g_1(t)_{t\in(0,T)}$ and $g_2(t)_{t\in(0,T)}$ be two Ricci flows on M which are smooth with respect to some differential structures on M. Assume furthermore that one can find $K > 0$ such that:

$$\forall (x,t) \in M \times (0,T) \quad K_{g_i(t)}(x) \geq -K$$

and that for $i = 1, 2$ the distances $d_{g_i(t)}$ uniformly converge to d as t goes to 0.

Then the two a priori different smooth structures on M agree and $g_1(t) = g_2(t)$ for $t \in (0,T)$.

This proposition is not a consequence of Theorem 0.5, but just requires a minor adjustment in its proof, which will be indicated in Section 2.

Proposition 0.7. – *Let $(M_1, g_1(t))_{t\in(0,T]}$ and $(M_2, g_2(t))_{t\in(0,T]}$ be two smooth Ricci flows such that for $i = 1, 2$ $(M_i, g_i(t))$ Gromov-Hausdorff converges to a compact Aleksandrov surface (X, d) with curvature bounded from below as t goes to 0. Assume furthermore that one can find $K > 0$ such that:*

$$\forall (x,t) \in M_i \times (0,T] \quad K_{g_i(t)}(x) \geq -K.$$

Then there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $g_2(t) = \varphi^* g_1(t)$.

Proof. – We just have to show that if $(M^2, g(t))_{t\in(0,T)}$ is a smooth Ricci flow on a surface M^2 such that for all $t \in (0,T)$ $K_{g(t)} \geq -K$ and such that $(M^2, g(t))$ Gromov-Hausdorff converges to (X, d) as t goes to 0, then (X, d) is the metric initial condition for the Ricci flow $(M^2, g(t))$.

Since the diameter and the volume are continuous with respect to Gromov-Hausdorff convergence with sectional curvature bounded from below, we have bounds on the diameter and the volume of $(M, g(t))$ which are independent of t. Thanks to the lower bound on the curvature, the upper bound on the diameter and the lower bound on the volume, the Bishop-Gromov inequality implies that we have some $v_0 > 0$ such that:

$$\forall t \in (0,T)\ \forall x \in M \quad \mathrm{vol}_{g(t)}(B_{g(t)}(x,1)) \geq v_0.$$