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MICROLOCAL DIRECT IMAGES OF SIMPLE SHEAVES WITH APPLICATIONS TO SYSTEMS WITH SIMPLE CHARACTERISTICS

BY

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RÉSUMÉ. — Dans ce papier, nous établissons un résultat d'hypoellipticité dans le cadre des problèmes aux limites microlocaux (à comparer aux résultats analogues de [SKK], [KS2]). Plus précisément, soit $\mathcal{M}$ un système d'équations microdifférentielles à caractéristiques simples sur une variété complexe X, et soit Λ_i $(i = 1, 2)$ un couple de sous-variétées Lagrangiennes réelles de T^*X. On note $\mathcal{C}_{\Lambda_i}$ les complexes des microfonctions associés. Si le couple (Λ_1, Λ_2) est « positif », nous prouvons l'injectivité du morphisme naturel de « restriction »

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_1})$$

entre les faisceaux de solutions, où j est le premier degré de cohomologie éventuellement non nul.

ABSTRACT. — In this paper we state a hypoellipticity result in the framework of microlocal boundary value problems (which has to be compared with the analogous results of [SKK], [KS2] at the interior). More precisely, let $\mathcal{M}$ be a system of microdifferential equations with simple characteristics on a complex manifold X, and let Λ_i $(i = 1, 2)$ be a pair of real Lagrangian submanifolds of T^*X. Denote by $\mathcal{C}_{\Lambda_i}$ the associated complexes of microfunctions. If the pair (Λ_1, Λ_2) is « positive », we prove the injectivity of the natural « restriction » morphism

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{\Lambda_1})$$

between solution sheaves, where j is the first possibly non-vanishing degree of the cohomology.

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Introduction

Let M be a real analytic manifold, let X be a complexification of M, and let $\mathcal{M}$ be a system of microdifferential equations with simple characteristics along a germ of smooth regular involutive submanifold V of the cotangent bundle T^*X to X.

A classical result of SATO, KAWAI and KASHIWARA [SKK] asserts that the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ of microfunction solutions to $\mathcal{M}$ has vanishing cohomology in degree smaller than $s^-(M,V)$, where $s^-(M,V)$ denotes the number of negative eigenvalues of the generalized Levi form of V with respect to $T^*_M X$.

The aim of this paper is to prove a similar result « up to the boundary ». More precisely, let $\Omega \subset M$ be an open subset with real analytic boundary S, and consider the complex $\mathcal{C}_\Omega$ of microfunctions at the boundary introduced by SCHAPIRA [S2] as a framework for the study of boundary value problems. Under some cleanness hypotheses, we get in this paper vanishing for the cohomology of the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_\Omega)$ in degree smaller then $s^-(M,V)$: this is a criterion of hypoellipticity for microlocal boundary value problems. Moreover, we give geometrical conditions that ensure the vanishing of the whole complex of solutions $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_\Omega)$.

Our method of proof is similar to that of KASHIWARA and SCHAPIRA [KS2] where, assuming the intersection $V \cap T^*_M X$ is clean and regular, they recover the above mentioned results of [S-K-K], and obtain new vanishing theorems. Let us examine their proof in some details.

First, they use a complex contact transformation χ which replaces $T^*_M X$ with the set Λ of exterior conormals to a strictly pseudoconvex open subset $U \subset X$, and which «straightens» V, i.e. which replaces V with $X \times_Y T^*Y$, where the fiber product is taken over a smooth holomorphic map $f: X \to Y$. A quantization of χ reduces $\mathcal{M}$ to a partial de Rham system, and interchanges the complex $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ with $\mu_{\partial U}(f^{-1}\mathcal{O}_Y)$, where $\mu_{\partial U}$ denotes the Sato microlocalization functor along ∂U. Finally, an adjunction formula reduces the study of the complex $\mu_{\partial U}(f^{-1}\mathcal{O}_Y)$ at $p \in (X \times_Y T^*Y) \cap \Lambda$ to the study of $\mu_{\partial U'}(\mathcal{O}_Y)$ at $f_\pi(p)$ (with $T^*_{\partial U'}Y = f_\pi(T^*_{\partial U}X \cap V)$). One is then reduced to the analysis of the microlocal direct image $f_!^{\mu,p} A_{\partial U}$ of the constant sheaf $A_{\partial U}$. The correspondence of cotangent bundles

$$T^*X \xleftarrow{{}^t f'} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y$$

associates to Λ a germ of Lagrangian manifold $f_\pi {}^t f'^{-1}(\Lambda) \subset T^*Y$ at $f_\pi(p)$. The sheaf $A_{\partial U}$ is simple along Λ with shift $\frac{1}{2}$, and it is thus possible to apply the following result of [KS3, Ch. 7]: if F is simple along Λ with shift d, then the microlocal direct image $f_!^{\mu,p} F$ is simple along $f_\pi {}^t f'^{-1}(\Lambda)$ with a shift which is calculated using generalized Levi forms. The vanishing theorems now easily follow.

Let $\Omega \subset M$ be an open subset with real analytic boundary S and consider the distinguished triangle $A_\Omega \to A_M \to A_S \xrightarrow{+1}$. In order to apply the above scheme of proof to boundary value problems, we have to deal simultaneously with the two Lagrangian manifolds $T^*_M X$ and $T^*_S X$. To this end, we first introduce a notion of «positive» pair of Lagrangian manifolds (to be compared with the one of [S1]) which allows us to give a microlocal meaning to the restriction morphism $A_M \to A_S$. In particular, if (Λ_1, Λ_2) is a pair of positive Lagrangian submanifolds of T^*X, there is a natural «restriction» morphism $\mathcal{C}_{\Lambda_2} \to \mathcal{C}_{\Lambda_1}$ between the associated complexes of generalized Sato microfunctions. Then, we give a criterion of faithfulness for the functor $f_!^{\mu,p}$ acting on simple sheaves. This allows us to state a very general result of injectivity for the the natural morphism

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{\Lambda_2}) \longrightarrow \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{\Lambda_1})$$

between solution sheaves to a simply characteristic system $\mathcal{M}$, where j is the first possibly non vanishing degree of the cohomology. In particular, this implies the criterion of hypoellipticity for microlocal boundary value problems we were looking for.

1. Symplectic geometry and Levi form

For the notions reviewed in this section, we refer to [SKK], [S1], [KS1], and [D'AZ2].

1.1. Generalized Levi forms. — Let (E,σ) be a complex symplectic vector space. A complex subspace $\rho \subset E$ is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if $\rho^\perp \supset \rho$ (resp. $\rho^\perp = \rho$, resp. $\rho^\perp \subset \rho$), where $(\,\cdot\,)^\perp$ denotes the orthogonal with respect to σ. A real subspace $\lambda \subset E$ is called $\mathbb{R}$-*Lagrangian* if it is Lagrangian in the underlying real symplectic vector space $(E^{\mathbb{R}},\sigma^{\mathbb{R}})$, where $\sigma^{\mathbb{R}} = 2\operatorname{Re}\sigma$.

If $\rho \subset E$ is isotropic, the form σ induces a symplectic structure on the space $\rho^\perp/\rho$, which is denoted by (E^ρ,σ^ρ). For a subset $\lambda \subset E$, let

$$\lambda^\rho = \big((\lambda \cap \rho^\perp) + \rho\big)/\rho \subset E^\rho$$

and recall that if λ is an $\mathbb{R}$-Lagrangian subspace of E then λ^ρ is an $\mathbb{R}$-Lagrangian subspace of E^ρ (as it follows from the formula $(\lambda^\rho)^\perp = (\lambda^\perp)^\rho$).

The following definition slightly generalizes that of [S1] and [KS2, Lemma 3.4].

Definition 1.1. — Let $\rho \subset E$ be isotropic, $\lambda \subset E$ be $\mathbb{R}$-Lagrangian, and set $\mu = \lambda \cap i\lambda$ (an isotropic space). The *generalized Levi form* $L_{\lambda/\rho}$ is the hermitian form on ρ^μ defined by setting for $v,w \in \rho^\mu$

$$L_{\lambda/\rho}(v,w) = \sigma^\mu(v,w^c),$$

where $(\,\cdot\,)^c$ denotes the conjugate with respect to the isomorphism $E^\mu \cong \mathbb{C} \otimes_{\mathbb{R}} \lambda^\mu$. We will denote by $s^\pm(\lambda,\rho)$ the numbers of positive (resp. negative) eigenvalues of $L_{\lambda/\rho}$.

The *inertia index* $\tau(\lambda_1,\lambda_2,\lambda_3)$ of three $\mathbb{R}$-Lagrangian subspaces $\lambda_j \subset E$, $j = 1,2,3$, is defined as the signature of the quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ given by

$$q(x_1,x_2,x_3) = \sigma^{\mathbb{R}}(x_1,x_2) + \sigma^{\mathbb{R}}(x_2,x_3) + \sigma^{\mathbb{R}}(x_3,x_1).$$

Let $\widetilde{\lambda}^\rho$ denote the $\mathbb{R}$-Lagrangian subspace $(\lambda \cap \rho^\perp) + \rho \subset E$.