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A UNIFORM DICHOTOMY FOR GENERIC SL(2, $\mathbb{R}$) COCYCLES OVER A MINIMAL BASE

by Artur Avila & Jairo Bochi

Abstract. — We consider continuous SL(2, $\mathbb{R}$)-cocycles over a minimal homeomorphism of a compact set K of finite dimension. We show that the generic cocycle either is uniformly hyperbolic or has uniform subexponential growth.

Résumé (*Une dichotomie uniforme pour des cocycles à valeurs dans* SL(2, $\mathbb{R}$) *au-dessus d'une dynamique minimale*)

On considère des cocycles continus à valeurs dans SL(2, $\mathbb{R}$) au-dessus d'un homéomorphisme minimal d'un ensemble compact de dimension finie. On montre que le cocycle générique soit est uniformément hyperbolique, soit possède une croissance sous-exponentielle uniforme.

1. Introduction

In this paper we will consider SL(2, $\mathbb{R}$)-valued cocycles over a minimal homeomorphism $f : K \to K$ of a compact set K. Such a cocycle can be defined

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as a pair (f, A) where $A : K \to \mathrm{SL}(2, \mathbb{R})$ is continuous. The cocycle acts on $K \times \mathbb{R}^2$ by $(x, y) \mapsto (f(x), A(x) \cdot y)$. The iterates of the cocycle are denoted $(f, A)^n = (f^n, A_n)$.

We say that (f, A) is *uniformly hyperbolic* if there exists $\varepsilon > 0$ and $N > 0$ such that $\|A_n(x)\| \geq \mathrm{e}^{\varepsilon n}$ for every $x \in K$, $n \geq N$. This is equivalent to the existence of a continuous invariant splitting $\mathbb{R}^2 = E^u(x) \oplus E^s(x)$ such that vectors in $E^s(x)$ are exponentially contracted by forward iteration and vectors in $E^u(x)$ are exponentially contracted by backwards iteration – see [7, Prop. 2].

We say that (f, A) has *uniform subexponential growth* if for every $\varepsilon > 0$ there exists $N > 0$ such that $\|A_n(x)\| \leq \mathrm{e}^{\varepsilon n}$ for every $x \in K$, $n \geq N$. This condition is equivalent to the vanishing of the Lyapunov exponent for all f-invariant probability measures (see Proposition 1 below). We recall that the Lyapunov exponent of the cocycle (f, A) with respect to an f-invariant probability measure μ is defined as

$$L(f, A, \mu) = \lim \frac{1}{n} \int_K \log \|A_n\| \mathrm{d}\mu.$$

We say that a compact set K has *finite dimension* if it is homeomorphic to a subset of some $\mathbb{R}^n$. For instance, compact subsets of manifolds (assumed as usual to be Hausdorff and second countable) have finite dimension. (For definitions of dimension and results concerning embedding in $\mathbb{R}^n$, see e.g. [6].)

THEOREM 1. — *Let $f : K \to K$ be a minimal homeomorphism of a compact set of finite dimension. Then for generic continuous $A : K \to \mathrm{SL}(2, \mathbb{R})$, either (f, A) is uniformly hyperbolic or (f, A) has uniform subexpoential growth.*

In the case where f is a minimal uniquely ergodic homeomorphism, Theorem 1 is contained in [1], which shows that if $f : K \to K$ is a homeomorphism, then for any ergodic f-invariant probability μ, the generic continuous $A : K \to \mathrm{SL}(2, \mathbb{R})$ is such that either $L(f, A, \mu) = 0$ or the restriction of (f, A) to the support of μ is uniformly hyperbolic. In general a minimal homeomorphism may admit uncountably many ergodic invariant probability measures, and in this paper we show how to treat them all at the same time. In the perturbation arguments, one often allows for loss of control in some sets, and this must be compensated by showing that such sets can be selected small. The notion of smallness needed for our purposes involves simultaneous conditions on all f-invariant probability measures.

It is unclear whether the hypothesis that K has finite dimension is actually necessary. The following example shows that the minimality hypothesis cannot be significantly weakened.

Example 1.1. — Let S^3 be identified with $\mathbb{C}^2_*/\mathbb{R}_+$ (here $\mathbb{C}^2_* = \mathbb{C}^2 \setminus \{(0,0)\}$ and $\mathbb{R}_+ = \{r \in \mathbb{R};\ r > 0\}$). Fix some $\alpha \in \mathbb{R}$ with $\alpha/2\pi$ irrational and let $f : S^3 \to S^3$ be given by

$$(z,w) \longmapsto \big(\mathrm{e}^{i\alpha} z,\, \mathrm{e}^{i\alpha}(z+w)\big) \bmod \mathbb{R}_+.$$

Notice that if $h : S^3 \to \mathbb{CP}^1$ with $h(z,w) = w/z$ is the usual Hopf fibration, and $g : \mathbb{CP}^1 \to \mathbb{CP}^1$ is $g(w) = w+1$, then $h \circ f = g \circ h$. So f has a unique minimal set, namely $S = h^{-1}(\infty)$, where f acts as a irrational rotation. Let $A : S^3 \to \mathrm{SL}(2,\mathbb{R})$ be any continuous map whose restriction to S is given by

$$(0, r\,\mathrm{e}^{i\theta}) \longmapsto \begin{pmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then $(f|S, A|S)$ is uniformly hyperbolic: the associated splitting is such that E^u and E^s are orthogonal and $E^u(0, r\,\mathrm{e}^{i\theta})$ is generated by $\binom{\cos\theta}{\sin\theta}$. This splitting is topologically nontrivial, and hence cannot be extended to the whole S^3. It follows that (f, A) is not uniformly hyperbolic and does not have uniform subexponential growth, and the same properties hold for any small perturbation of A.

Using ideas from [2], [5], one can adapt the arguments of this paper to deal with $\mathrm{GL}(d,\mathbb{R})$-valued cocycles. The conclusion is that *for a generic continuous $A : K \to \mathrm{GL}(d,\mathbb{R})$ and for every f-invariant probability measure μ, the Oseledets splitting relative to μ coincides almost everywhere with the finest dominated splitting of (f, A).*

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2. Uniform subexponential growth

Here we prove an equivalence stated at the introduction:

Proposition 1. — *Let $f : K \to K$ be homeomorphism of a compact set K. and $A : K \to \mathrm{SL}(2,\mathbb{R})$ be a continuous map. Then the following are equivalent:*

(a) *(f, A) has uniform subexponential growth: for every $\varepsilon > 0$ there exists $N > 0$ such that $\|A_n(x)\| \le \mathrm{e}^{\varepsilon n}$ for every $x \in K$, $n \ge N$;*

(b) *for every $\varepsilon > 0$ there exists $n > 0$ such that $\|A_n(x)\| \le \mathrm{e}^{\varepsilon n}$ for every $x \in K$;*

(c) *$L(f, A, \mu) = 0$ for every f-invariant probability μ.*

In the case that f is uniquely ergodic, the proposition follows from [4, Thm. 1].

Proof of the proposition. — (a) $\Rightarrow$ (b) is trivial; (b) $\Rightarrow$ (c) follows from the fact that $L(f,A,\mu) = \inf_n n^{-1} \int_K \log \|A_n\| \mathrm{d}\mu$.

We are left to prove (c) $\Rightarrow$ (a). Assume that (a) does not hold. Then there exists a sequence $x_k \in K$, $n_k \to \infty$ such that $\|A_{n_k}(x_k)\| \geq \mathrm{e}^{\varepsilon n_k}$. Let $\mu_k = {n_k}^{-1} \sum_{j=0}^{n_k-1} \delta_{f^j(x_k)}$. Passing to a subsequence, we may assume that μ_k converges to μ, which is f-invariant. We claim that $L(f,A,\mu) \geq \varepsilon$.

Let $\delta > 0$ and $s \in \mathbb{N}$ be fixed. It is enough to show that

$$\int \log \|A_s\| \mathrm{d}\mu \geq (\varepsilon - \delta)s.$$

Let $m_k = \lfloor n_k/s \rfloor$. Let $\nu_k = (sm_k)^{-1} \sum_{j=0}^{sm_k-1} \delta_{f^j(x_k)}$. Notice that $\nu_k \to \mu$. It is clear that if k is large then $\|A_{sm_k}(f^i(x_k))\| \geq \mathrm{e}^{(\varepsilon-\delta)sm_k}$ for $0 \leq i \leq s-1$. Then

$$\begin{aligned}\int \log \|A_s\| d\nu_k &= \frac{1}{sm_k} \sum_{i=0}^{s-1} \sum_{j=0}^{m_k-1} \log \left\|A_s\left(f^{js+i}(x_k)\right)\right\| \\ &\geq \frac{1}{sm_k} \sum_{i=0}^{s-1} \log \left\|A_{sm_k}\left(f^i(x_k)\right)\right\| \geq s(\varepsilon - \delta).\end{aligned}$$

The result follows. $\square$

3. Perturbation along segments of orbits

In this section we assume that $f : K \to K$ is minimal with no periodic orbits and $A : K \to \mathrm{SL}(2,\mathbb{R})$ is a continuous map such that (f, A) is not uniformly hyperbolic, but there exists an f-invariant measure such that $L(f,A,\mu) > 0$. The aim here is to establish Lemma 2 (see below).

We begin with an adaptation of Lemma 3.4 from [5]:

LEMMA 1. — *For every $\varepsilon > 0$, there exists a non-empty open set $W \subset K$, and $m \in \mathbb{N}$ such that:*

- *W, $f(W)$, …, $f^{m-1}(W)$ are disjoint;*
- *for all $x \in W$ and any non-zero vectors $\boldsymbol{v}$, $\boldsymbol{w}$, there exists $M_0, \dots, M_{m-1}$ in $\mathrm{SL}(2,\mathbb{R})$ such that $\|M_j - A(f^j(x))\| < \varepsilon$ and $M_{m-1} \cdots M_0(\boldsymbol{v})$ is collinear to $\boldsymbol{w}$.*