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LOGARITHMIC BUNDLES OF DEFORMED WEYL ARRANGEMENTS OF TYPE A_2

by Takuro Abe, Daniele Faenzi & Jean Vallès

Abstract. — We consider deformations of the Weyl arrangement of type A_2, which include the extended Shi and Catalan arrangements. These last ones are well-known to be free. We study their sheaves of logarithmic vector fields in all other cases, and show that they are Steiner bundles. Also, we determine explicitly their unstable lines. As a corollary, some counter-examples to the shift isomorphism problem are given.

Résumé (*Fibrés logarithmiques des arrangements de Weyl déformés de type A_2*)
Nous considérons des déformations des arrangements de Weyl de type A_2, déformations dont les arrangements de Shi et de Catalan forment une classe particulière. Il est bien connu que ces derniers sont libres. Nous étudions les faisceaux de champs de vecteurs logarithmiques des autres arrangements déformés et montrons qu'ils sont des fibrés de Steiner. Nous déterminons explicitement leurs droites instables. Comme corollaire, des contres exemples du problème appelé « shift isomorphism » sont donnés.

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Introduction

Let Φ be an irreducible crystallographic root system in Euclidean space $V \simeq \mathbb{R}^m$, let $\Phi^+ \subset \Phi$ be the positive roots, and let η be the Coxeter number of Φ. Let $x_1, \ldots, x_m$ be coordinates of V, set $S = \mathbb{R}[x_0, \ldots, x_m]$, and denote by $\mathrm{Der}(S)$ the free S-module of derivations of S, generated by the partial derivatives $\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_m}$. For $s \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define the hyperplanes:

$$H_{\alpha,s} = \{x \in \mathbb{P}^m \mid \alpha(x_1, \ldots, x_m) = sx_0\} \subset \mathbb{P}^m.$$

Fix integers $k, j \geq 0$, and define the (cone over the) *deformation of the Weyl arrangement of type* Φ:

$$\mathscr{A}_\Phi^{[-j,k+j]} = \{x_0 = 0\} \cup \{H_{\alpha,s} \mid \alpha \in \Phi^+, -j \leq s \leq k+j\}.$$

The combinatorics, topology and algebra of $\mathscr{A} = \mathscr{A}_\Phi^{[-j,k+j]}$ have been studied by several authors, for instance by Postnikov and Stanley in [10], by Athanasiadis in [3], by Edelman and Reiner in [5], and by Yoshinaga in [14], especially when $k \in \{0,1\}$. In particular, the freeness of $\mathscr{A}$ when $k = 0,1$ was conjectured by Edelman and Reiner and proved in [14] by Yoshinaga. By *freeness* here we mean freeness of logarithmic derivation module of $\mathscr{A}$:

$$D_0(\mathscr{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(f_{j,k}) = 0\},$$

where $f_{j,k}$ is the form of degree $n = |\mathscr{A}|$ given as product of linear forms defining the hyperplanes of $\mathscr{A}$. Equivalently, freeness means splitting of the the sheafification $T_\mathscr{A}$ of $D_0(\mathscr{A})$. This is a reflexive sheaf of rank m called *logarithmic sheaf*. It can also be defined as the kernel of the Jacobian map:

$$\mathscr{O}_{\mathbb{P}^m}^{m+1} \xrightarrow{\nabla(f_{j,k})} \mathscr{O}_{\mathbb{P}^m}(n-1).$$

In spite of the good knowledge of $T_\mathscr{A}$ for $k \in \{0,1\}$, almost nothing is known about $T_\mathscr{A}$ for $k \geq 2$, not even for A_2. For example, setting $\mathscr{B} = \mathscr{A}_\Phi^{[-j-1,k+j+1]}$, the *shift isomorphism problem*, cf. [15, Remark 3.7] asks whether there is an isomorphism:

$$T_\mathscr{A} \simeq T_\mathscr{B}(\eta). \tag{1}$$

Another question is the *shifted dual isomorphism problem*, to the effect that:

$$T_\mathscr{A} \simeq T_\mathscr{A}^\vee(-\eta(k+2j+1)). \tag{2}$$

These isomorphisms hold when $k = 0,1$ by [14]. However, even equality of characteristic polynomials (i.e., of Chern classes) of these sheaves is unknown in general: this is the so-called "functional equation" conjecture of [10], cf. also [15, Conjecture 3.4 and 3.5]. However the roots of the characteristic polynomial should have real part $\eta(k+2j+1)/2$ by the "Riemann hypothesis" of [10], verified for Φ of type A, B, C, D in [2].

In this paper, we are most interested in the case $\Phi = A_2$. We switch to the notation (z, x, y) rather than (x_0, x_1, x_2), and we fix $\mathscr{A} = \mathscr{A}_{A_2}^{[-j,k+j]}$. We have $\eta = 3$. In this case $T_{\mathscr{A}}$ is locally free (a vector bundle) of rank 2, and the lines of $\mathscr{A}$ are defined by vanishing of the form:

$$f_{j,k} = z \prod_{-j \le s \le k+j} (x - sz)(y - sz)(y + x - sz).$$

Concerning resolutions, our main theorem is the following.

THEOREM 1. — *For any $k \ge 2$ and $j \ge 0$, there is a resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1)^{k-1} \to \mathscr{O}_{\mathbb{P}^2}^{k+1} \to T_{\mathscr{A}}(2k + 1 + 3j) \to 0.$$

In particular, $T_{\mathscr{A}}(2k + 1 + 3j)$ is a Steiner bundle.

By *Steiner bundle* here we mean a vector bundle whose resolution is given by a matrix of linear forms. This agrees and gives a new interpretation of the following formulas, easily obtained for instance counting multiple points and using [7, Remark 2.2]:

$$c_1(T_{\mathscr{A}}(2k + 3j + 1)) = k - 1, \qquad c_2(T_{\mathscr{A}}(2k + 3j + 1)) = \tfrac{k(k-1)}{2}.$$

Since $T_{\mathscr{A}}(2k + 1 + 3j)$ is a Steiner bundle, for any line $L \subset \mathbb{P}^2$, by restriction onto L we get a surjective map:

$$\mathscr{O}_L^{k+1} \longrightarrow T_{\mathscr{A}}(2k + 1 + 3j)|_L.$$

This implies that $T_{\mathscr{A}}(2k+1+3j)|_L = \mathscr{O}_L(a) \oplus \mathscr{O}_L(k-1-a)$ with $0 \le a \le k-1$. When $a = 0$ or $a = k - 1$ the number $|k - 1 - 2a|$ is as large as possible. This justifies the next definition, cf. [11, Page 508] or [6, Definition 2.1].

DEFINITION 1. — Let $k \ge 2$ and E be a Steiner bundle defined by:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1)^{k-1} \longrightarrow \mathscr{O}_{\mathbb{P}^2}^{k+1} \longrightarrow E \longrightarrow 0.$$

A line L such that $E|_L = \mathscr{O}_L \oplus \mathscr{O}_L(k - 1)$ or equivalently $H^0(\mathbb{P}^2, E^\vee|_L) \ne 0$ or equivalently $H^1(L, E|_L(-2)) \ne 0$ is called *unstable*. The set of such lines is denoted by $W(E)$, it is naturally a subscheme of $\check{\mathbb{P}}^2$. These unstable lines were first called *superjumping lines* in [4].

Our next result, tightly related with Theorem 1, deals with the set of unstable lines of $T_{\mathscr{A}}$, which we can determine explicitly. The figure shows them in case $j = 0$ and $k = 3$ or $k = 4$, the thick orange lines being unstable (the solid ones lie in the arrangement, the dashed ones don't).