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DISTRIBUTION OF IRRATIONAL ZETA VALUES

by Stéphane Fischler

Abstract. — In this paper we refine the Ball-Rivoal theorem by proving that for any odd integer a sufficiently large in terms of $\varepsilon > 0$, there exist $\lfloor \frac{(1-\varepsilon)\log a}{1+\log 2} \rfloor$ odd integers s between 3 and a, with distance at least a^{ε} from one another, at which the Riemann zeta function takes $\mathbb{Q}$-linearly independent values. As a consequence, if there are very few odd integers s such that $\zeta(s)$ is irrational, then they are rather evenly distributed.

The proof involves series of hypergeometric type, a trick to apply the saddle point method with parameters, and the generalization to vectors of Nesterenko's linear independence criterion.

Résumé (*Répartition des valeurs irrationnelles de zêta*). — Dans cet article on raffine le théorème de Ball-Rivoal en démontrant que pour tout entier impair a suffisamment grand en fonction de $\varepsilon > 0$, il existe $\lfloor \frac{(1-\varepsilon)\log a}{1+\log 2} \rfloor$ entiers impairs s entre 3 et a, écartés les uns des autres d'au moins a^{ε}, en lesquels la fonction zêta de Riemann prend des valeurs $\mathbb{Q}$-linéairement indépendantes. En conséquence, si il y a très peu d'entiers impairs s tels que $\zeta(s)$ soit irrationnel, alors ils sont assez bien répartis.

La preuve utilise des séries de type hypergéométrique, une astuce pour appliquer la méthode du col en présence de paramètres, et la généralisation aux vecteurs du critère d'indépendance linéaire de Nesterenko.

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1. Introduction

Conjecturally, all values of the Riemann zeta function at odd integers $s \geq 3$ are irrational, and together with 1 they are linearly independent over the rationals. However very few results are known in this direction. After Apéry's breakthrough, namely the proof [1] that $\zeta(3)$ is irrational, the next major result is due to Ball-Rivoal ([2], [11]):

THEOREM 1.1 (Ball-Rivoal). — *Let $\varepsilon > 0$, and a be an odd integer sufficiently large with respect to ε. Then the $\mathbb{Q}$-vector space*

$$\mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \ldots, \zeta(a)) \tag{1.1}$$

has dimension at least $\frac{1-\varepsilon}{1+\log 2}\log(a)$.

Except when a is bounded, this is the only known linear independence result on the values $\zeta(s)$ for odd $s \leq a$. Trying to find integers s such that $\zeta(s)$ is irrational, the following result of Zudilin (Theorem 0.2 of [12]) has also to be mentioned.

THEOREM 1.2 (Zudilin). — *For any odd integer $d \geq 1$, at least one of the numbers*

$$\zeta(d+2), \quad \zeta(d+4), \quad \zeta(d+6), \quad \ldots, \quad \zeta(8d-1)$$

is irrational.

The purpose of the present paper is to prove results on the distribution of (provably) irrational (or linearly independent) zeta values. For instance, given a large odd integer a, Theorems 1.1 and 1.2 do not exclude the possibility that 1, $\zeta(3)$, $\zeta(5), \ldots, \zeta(N)$ are $\mathbb{Q}$-linearly independent, with $N = 2\lfloor\frac{\log a}{1+\log 2}\rfloor$, and $\zeta(N+2)$, $\zeta(N+4), \ldots, \zeta(a)$ are all rational multiples of $\zeta(3)$. More generally, there might exist a few small blocks of consecutive odd integers among which one has to take the integers $s \leq a$ so that the values $\zeta(s)$ make up a basis of the $\mathbb{Q}$-vector space (1.1), for instance $c\log a$ blocks of length some fixed power of $\log a$, with $c < 1/(1+\log 2)$. We prove that this cannot happen for a sufficiently large, as the following result shows.

THEOREM 1.3. — *Let $\varepsilon > 0$, and $a \geq d \geq 1$ be such that $0 < \varepsilon \leq 1/20$ and $a \geq \varepsilon^{-12/\varepsilon}d$. Then there exist odd integers $\sigma_1, \ldots, \sigma_N$ between d and a, with $N = \lfloor\frac{1-\varepsilon}{1+\log 2}\log(a/d)\rfloor$, such that:*

- *1, $\zeta(\sigma_1), \ldots, \zeta(\sigma_N)$ are linearly independent over the rationals.*
- *For any $i \neq j$, we have $|\sigma_i - \sigma_j| > d$.*

Taking $d = a^\varepsilon$ in this result, we obtain Theorem 1.1 with two additional properties: linearly independent zeta values with distance at least a^ε from one another, and an explicit value $a(\varepsilon)$ such that the conclusion of Theorem 1.1 holds for any $a \geq a(\varepsilon)$. The latter could have been derived from Ball-Rivoal's proof ([2], [11]), whereas the former is the central new result of the present paper.

Coming back to arbitrary values of d, one may weaken the conclusion $|\sigma_i - \sigma_j| > d$ of Theorem 1.3 to $\sigma_i > d$, discarding at most one zeta value $\zeta(\sigma_j)$. This yields the following corollary, in which for simplicity we omit the explicit relations of Theorem 1.3 on ε, d, a.

COROLLARY 1.4. — *Let $\varepsilon > 0$. Let $a \geq 3$ and $d \geq 1$ be odd integers such that a/d is sufficiently large (in terms of ε). Then*

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\zeta(d), \zeta(d+2), \zeta(d+4), \dots, \zeta(a)) \geq \frac{(1-\varepsilon)\log(a/d)}{1+\log 2}.$$

Moving now to bounded values of a, Ball and Rivoal have proved ([2], [11]) that (1.1) has dimension at least 3 for $a = 169$. This numerical value has been improved to 145 by Zudilin [12], and to 139 in [5]. We obtain the following result in the spirit of Theorem 1.2 and Corollary 1.4.

THEOREM 1.5. — *For any odd integer $d \geq 1$ there exist odd integers σ_1, σ_2 with*

$$d + 2 \leq \sigma_1 < \sigma_2 \leq 151\, d, \qquad \sigma_2 > \sigma_1 + 6 \cdot 10^{-6}\, d,$$

such that 1, $\zeta(\sigma_1)$ and $\zeta(\sigma_2)$ are $\mathbb{Q}$-linearly independent.

This result is new for any $d \geq 3$, even if $\sigma_2 > \sigma_1 + 6 \cdot 10^{-6}\, d$ is omitted. The numerical value 151 (instead of 145 or 139) comes from the fact that some estimates are slightly worse when d is large than for $d = 1$.

Now let us move from linear independence to irrationality of zeta values. The Ball-Rivoal theorem yields an increasing sequence $(u_i)_{i \geq 1}$ of odd integers such that $\zeta(u_i) \notin \mathbb{Q}$ for any i, and $\limsup u_i^{1/i} \leq 2e$; for instance it is enough to denote by u_i the i-th odd integer $s \geq 3$ such that $\zeta(s) \notin \mathbb{Q}$. The existence of such a sequence with $\lim u_i^{1/i} = 2e$ can be deduced from Corollary 1.4 (by following the proof of Corollary 1.6 below). Actually, using Theorem 1.3 we obtain the following result, in which the odd integers u_i are quite distant from one another.

COROLLARY 1.6. — *Let ε be a positive real number such that $\varepsilon \leq 1/20$; put $\eta = \varepsilon^{15/\varepsilon}$. Then there exists an increasing sequence $(u_i)_{i \geq 1}$ of odd integers, depending only on ε, with the following properties:*

- *For any $i \geq 1$, $\zeta(u_i)$ is an irrational number.*
- *For any $i \geq 1$, we have $u_{i+1}/u_i > 1 + \eta$.*
- *For any $i \geq 1$, we have $\eta(2e)^{(1+\varepsilon)i} < u_i < \eta^{-1}(2e)^{(1+\varepsilon)i}$.*
- *For any $a \geq \eta^{-1/\varepsilon}$ we have $u_N \leq a$, where N is the integer part of $\frac{1-2\varepsilon}{1+\log 2} \log a$.*