Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON DEFORMATIONS OF THE SPECTRUM OF A FINSLER-LAPLACIAN THAT PRESERVE THE LENGTH SPECTRUM

Thomas Barthelmé

Tome 145
Fascicule 3

2017

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 421-448

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 145, septembre 2017

Diffusion

Maison de la SMF - Case 916 - Luminy - 13288 Marseille Cedex 9 - France
christian.smf@cirm-math.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
bullsmf@ihp.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
145 (3), 2017, p. 421-448

ON DEFORMATIONS OF THE SPECTRUM OF A FINSLER-LAPLACIAN THAT PRESERVE THE LENGTH SPECTRUM

BY THOMAS BARTHELMÉ

ABSTRACT. — The main result of this article is the construction of non-reversible Finsler metrics in negative curvature such that $4\lambda_1 > h^2$, where λ_1 is the bottom of the L^2-spectrum of a previously defined Finsler-Laplacian and h the topological entropy of the flow. This gives a counter-example to a classical inequality in Riemannian geometry. We also show that the spectrum of that Finsler-Laplacian can detect changes in the Finsler metric that the marked length spectrum cannot.

RÉSUMÉ (*Sur des déformations du spectre d'un opérateur de Finsler-Laplace préservant le spectre des longueurs*). — Le résultat principal de cet article est la construction d'une famille de métriques de Finsler, non-réversible, en courbure négative satisfaisant $4\lambda_1 > h^2$, où λ_1 est le bas du spectre L^2 d'un laplacien en géométrie de Finsler et h est l'entropie topologique du flot géodésique. Ce résultat fournit un contre-exemple, pour les métriques de Finsler, à une inégalité classique de géométrie riemannienne. Nous montrons également que le spectre de ce laplacien détecte certains changements de la métrique qui sont invisible pour le spectre des longueurs.

Finsler metrics have a long history of producing quite different results from what one might expect from Riemannian metrics. Among the general classes of Finsler metrics from which surprises can arise are non-reversible Finsler metrics. Non-reversible Finsler metrics are defined by considering norms which are not symmetric with respect to 0, or in other words, such that their unit balls in each

Texte reçu le 25 juin 2015, accepté le 23 mai 2016.

THOMAS BARTHELMÉ, Pennsylvania State University, State College, PA, USA • Queen's University, Kingston, ON, Canada • *E-mail :* thomas.barthelme@queensu.ca • *Url :* https://sites.google.com/site/thomasbarthelme

Mathematical subject classification (2010). — 58J60, 53C60.

Key words and phrases. — Finsler-Laplacian, topological entropy, length spectrum.

doi:10.24033/bsmf.2743

tangent space are convex sets that contain, but are not centered at, the origin. One of the most striking surprise that arose from non-reversible Finsler metrics was the construction by Katok [18] in 1973 of a metric on the sphere with only 2 periodic geodesics (these metrics are now called Katok-Ziller metrics as they have been thoroughly studied by Ziller in [30]). The Katok-Ziller metrics turned out to be Randers metric, i.e., metrics of the form $F = \sqrt{g} + \beta$ where g is a Riemannian metric and β is a one-form.

We are interested in this article in the Finsler-Laplacian spectrum of Randers metrics, or more generally Finsler metrics of the form $F = \bar{F} + \beta$ where $\bar{F}$ is a reversible Finsler metric and β a one-form. The operator we consider is the Finsler-Laplacian introduced in [1, 2]. Since this operator was thought of by Jean-Pierre Bourguignon and Patrick Foulon, who then suggested it to me, we will henceforth call this operator the BF-Laplacian (we recall its construction in Section 1 below).

While considering this operator, we already had some surprising results in the non-reversible case: In [3], Colbois and myself showed that, for any surface S and any *reversible* Finsler metric $\bar{F}$, there exists a *uniform* constant K (depending only on the topology of S) such that $\lambda_1(\bar{F}) \operatorname{vol}\left(S, \bar{F}\right) \leq K$. This result is just a generalization of a classical Riemannian result [21]. But we also proved that, for any $C > 0$ and any surface S, there exists a Randers metric F such that $\lambda_1(F) \operatorname{vol}(S, F) \geq C$. So, allowing a metric to be non-reversible can yield examples of metrics with a λ_1 much bigger than it should be.

We will construct here examples of non-reversible metrics that yield two more surprises. The first with respect to a presumed link between marked length spectrum and the spectrum of the Laplacian and the second with respect to the link between the bottom of the spectrum and the topological entropy of the geodesic flow.

The length spectrum of a metric is defined as the set of lengths of closed geodesics counted with multiplicity. Two manifolds are said to have the same *marked* length spectrum if there is an isomorphism of their fundamental group such that corresponding free homotopy classes contain closed geodesics of the same length. The link between Laplacian spectrum and length, or marked length, spectrum has been intensively studied in Riemannian geometry. Generically, the length spectrum of a Riemannian manifold is determined by the Laplacian spectrum (Colin de Verdière [29]). In some specific cases, the notions of marked length spectrum and Laplacian spectrum are in fact equivalent in the sense that one determines the other and vice versa. Among the manifolds that verifies this are for instance flat tori (see for instance [14]), manifolds of negative curvature (Otal [25] and Croke [7] since in that case equality of the marked length spectrum implies isometry), and some types of nilmanifolds (see [9], and it is in fact conjectured to be true for all nilmanifolds [15]).

Examples of Riemannian manifolds with the same length spectrum but not isospectral exists however, but they are quite exceptional. One such example is comparing two different Zoll surfaces (i.e., a metric on the sphere such that all its geodesics are closed and of length 2π, see [16]).

Non-reversible Finsler metrics give a very contrasted picture: for *any* (reversible) metric on *any* manifold, we can construct a non-reversible metric with the same marked length spectrum and different spectra:

THEOREM 1. — *Let $\bar{F}$ be a reversible Finsler metric on a manifold M, we denote by $\bar{F}^*$ the dual metric. Let $F = \bar{F} + \beta$, where β is an* exact *1-form on M, not identically zero, such that $\bar{F}^*(\beta) < 1$. Then $\bar{F}$ and F have the same marked length spectrum and the same volume, but, for each k, $\lambda_k(F) > \lambda_k(\bar{F})$.*

Note that the condition on the norm of β is only there to insure that the metric F is still a Finsler metric.

Saying that this result is really surprising might be a bit of a stretch. Indeed, there exist infinite-dimensional families of Finsler metrics that share the same marked length spectrum, so finding some metrics with different spectra should not be too hard. But on the other hand, infinitely many Finsler metrics should also share the same BF-Laplacian (see [1, 2]), which makes the existence of the above examples not completely obvious.

Moreover, the main interest of this result is what it suggests about the BF-Laplacian: this type of transformation of a reversible metric by an exact form does not change the metric, or the geodesic flow a lot. Indeed, the new geodesic flow is a time change of the old that do not change the length of any closed geodesic. In fact such a time-change is a trivial time change in the terminology of [20], i.e., it is a time change such that the two flows are smoothly conjugate (and this is all due to the fact that β is taken to be exact, see Lemma 11). So the length spectrum is not subtle enough to pick up this change, nor is the dynamics of the geodesic flow. But what the above result shows is that the BF-Laplacian do detect such variations, which could make it a more powerful tool in some situations.

If we work a bit more, we can obtain some even more surprising examples:

THEOREM 2. — *Let g_0 be the flat metric on the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $F_{\varepsilon,t} = \sqrt{g_0} + t dh_\varepsilon$ be a Randers metric, where h_ε is a well chosen function such that, almost everywhere, ∇h_ε tends to a unit vector of irrational slope. Then, for all ε, t, $(\mathbb{T}^2, F_{\varepsilon,t})$ have the same volume, the same geodesic flow up to a (trivial) time-change and the same marked length spectrum as $(\mathbb{T}^2, g_0)$, but*

$$\lim_{(\varepsilon,t)\to(0,1)} \lambda_1(F_{\varepsilon,t}) = +\infty.$$

The family of functions h_ε are given explicitly in Section 3.2. This result is an improvement on the example on the torus constructed in [3]. First of all because this new example preserves the marked length spectrum. But also,