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QUADRATIC OPERATOR PENCILS ASSOCIATED WITH THE CONSERVATIVE CAMASSA-HOLM FLOW

BY JONATHAN ECKHARDT & ALEKSEY KOSTENKO

ABSTRACT. — We discuss direct and inverse spectral theory for a Sturm-Liouville type problem with a quadratic dependence on the eigenvalue parameter,

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f,$$

which arises as the isospectral problem for the conservative Camassa-Holm flow. In order to be able to treat rather irregular coefficients (that is, when ω is a real-valued Borel measure on $\mathbb{R}$ and υ is a non-negative Borel measure on $\mathbb{R}$), we employ a novel approach to study this spectral problem. In particular, we provide basic self-adjointness results for realizations in suitable Hilbert spaces, develop (singular) Weyl-Titchmarsh theory and prove several basic inverse uniqueness theorems for this spectral problem.

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Résumé (*Pinceaux quadratiques d'opérateurs associés au flot Camassa-Holm conservateur*). — Nous discutons la théorie spectrale directe et inverse pour un problème tapez Sturm-Liouville avec une dépendance quadratique du paramètre valeur propre,

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f,$$

qui se pose le problème isospectral pour le flot Camassa-Holm conservateur. Afin d'être capable de traiter des coefficients plutôt irréguliers (qui est, quand ω est une mesure de Borel de valeur réelle sur $\mathbb{R}$ et υ est une mesure de Borel non-négative sur $\mathbb{R}$), nous employons une nouvelle approche pour étudier ce problème spectral. En particulier, nous fournissons des résultats auto-adjointness de base pour des réalisations dans des espaces de Hilbert appropriés, développons théorie Weyl-Titchmarsh (singulier) et prouvons plusieurs théorèmes de base d'unicité inverse pour ce problème spectral.

1. Introduction

The principal purpose of the present article is to discuss direct and inverse spectral theory for a Sturm-Liouville type problem of the form

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f, \tag{1.1}$$

where ω is a real-valued Borel measure on $\mathbb{R}$, υ is a non-negative Borel measure on $\mathbb{R}$ and z is a complex spectral parameter. The significance of this rather specific spectral problem stems from the fact that it arises as the isospectral problem of a particular completely integrable nonlinear wave equation. More precisely, it has been identified as an isospectral problem for the two-component Camassa-Holm system [12, 32] and it turned out recently [22] that it also serves as an isospectral problem for global conservative solutions of the Camassa-Holm equation [10, 29, 31]. Regarding further information about the Camassa-Holm equation, we only refer to a brief selection of articles [8, 11, 13, 14, 16, 17, 47, 48].

Inverse spectral and scattering theory for the Sturm-Liouville type problem (1.1) is of peculiar interest for solving the Cauchy problem for the Camassa-Holm equation and its two-component generalization. Since the coefficient ω is allowed to change sign and because of the presence of the measure υ, spectral theory for (1.1) is outside of most standard theory for Sturm-Liouville problems and requires distinct methods to deal with it. In particular, direct and inverse spectral theory for (1.1) is still not sufficiently developed for applications to the Camassa-Holm flow (but see [3, 5, 6, 7, 13, 15, 21, 22, 23, 27]). Moreover, except for [22], all of these references only deal with the case when the measure υ is not present at all. However, let us also mention that problems similar to (1.1) have been studied in [42, 43, 44, 45] in the context of indefinite strings, where the authors dealt with the spectral problem in a Krein space setting.

In this article, we provide a thorough operator theoretic framework to treat the spectral problem (1.1) which will serve as a solid basis for further investigations on the integrability of the conservative Camassa-Holm flow. More precisely, we will provide basic self-adjointness results for realizations of this spectral problem (on an interval $J \subseteq \mathbb{R}$) in Hilbert spaces of the form

$$\mathcal{H}(J) = H^1(J) \times L^2(J; \upsilon), \tag{1.2}$$

equipped with a suitable scalar product. These self-adjoint realizations are mostly of an auxiliary nature, whereas the more convenient objects seem to be associated quadratic operator pencils in $H^1(J)$ which will be introduced next. We will also introduce (singular) Weyl-Titchmarsh functions, which are basic objects of spectral theory for Sturm-Liouville problems (for further information on singular Weyl-Titchmarsh functions we refer to [25, 26, 28, 35, 39, 40]). All this will be done for the cases of bounded intervals (in Section 3), semi-axes (in Section 4) and the whole line (in Section 5) separately. Even though it could be done at once in principle, we decided to present all these cases separately for the sake of clearness and to avoid distracting case differentiations and awkward notation. Since the whole line case is of particular importance for applications to the Camassa-Holm flow, we will furthermore introduce a spectral measure and a spectral transformation in this case as well. In the final Section 6, we will provide several basic inverse uniqueness theorems for the spectral problem (1.1) following [21]. More precisely, we will provide Borg-Marchenko type uniqueness results for the spectral problem on semi-axes as well as some uniqueness results for the whole line.

Although our main motivation lies in applications to the conservative Camassa-Holm flow, we think that the present article is also of interest to a wider audience since it provides a new way to treat Sturm-Liouville type problems with a quadratic dependence on the spectral parameter. The theory developed in this article for example also works for more general problems of the form

$$-f'' + \chi f = z\,\omega f + z^2 \upsilon f, \tag{1.3}$$

where χ is a non-negative Borel measure on $\mathbb{R}$. In this context, let us also mention that the spectral problem (1.1) can be transformed via a Liouville transform to a Schrödinger spectral problem with an energy dependent potential

$$-f'' + qf + z\,pf = z^2 f, \tag{1.4}$$

provided that the measures ω and υ are sufficiently smooth and positive. Spectral problems (1.4) arise in various contexts and we just mention [33, 34, 38, 52] for further information and references. However, for our applications this transformation is not possible since we have to allow more general coefficients ω and υ.