Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

CUTTING ARCS FOR TORUS LINKS AND TREES

Filip Misev

Tome 145
Fascicule 3

2017

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 575-602

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 3, tome 145, septembre 2017

Comité de rédaction

Christine BACHOC
Emmanuel BREUILLARD
Yann BUGEAUD
Jean-François DAT
Charles FAVRE
Marc HERZLICH
Raphaël KRIKORIAN
Laurent MANIVEL
Julien MARCHÉ
Kieran O'GRADY
Emmanuel RUSS
Christophe SABOT
Wilhelm SCHLAG

Pascal HUBERT (Dir.)

Diffusion

Maison de la SMF - Case 916 - Luminy - 13288 Marseille Cedex 9 - France
`christian.smf@cirm-math.fr`

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
`bullsmf@ihp.fr` • `smf.emath.fr`

© *Société Mathématique de France 2017*

Tous droits réservés (article L 122–4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335–2 et suivants du CPI.

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
145 (3), 2017, p. 575-602

CUTTING ARCS FOR TORUS LINKS AND TREES

BY FILIP MISEV

ABSTRACT. — Among all torus links, we characterise those arising as links of simple plane curve singularities by the property that their fibre surfaces admit only a finite number of cutting arcs that preserve fibredness. The same property allows a characterisation of Coxeter-Dynkin trees (i.e., A_n, D_n, E_6, E_7 and E_8) among all positive tree-like Hopf plumbings.

RÉSUMÉ (*Découpages d'entrelacs toriques et de plombages positifs arborescents*). — Parmi les entrelacs toriques, nous caractérisons ceux qui apparaissent comme entrelacs d'une singularité simple d'une courbe plane par la propriété que leurs surfaces fibres n'admettent qu'un nombre fini d'arcs de découpage qui préservent la fibration. La même propriété permet une caractérisation des arbres de Coxeter-Dynkin (i.e., A_n, D_n, E_6, E_7 et E_8) parmi tous les plombages positifs arborescents.

1. Introduction

A *fibred link* is a link $L \subset S^3$ such that $S^3 \setminus L$ fibers over the circle, and where each fibre is the interior of a Seifert surface S for L in S^3. Cutting S along a properly embedded interval α (an *arc* for short) results in another Seifert surface S' for another link $\partial S' = L'$. If L' is again a fibred link with fibre S', we say that α *preserves fibredness.* For example, α could be the spanning

Texte reçu le 1^er septembre 2015, modifié le 28 janvier 2016, accepté le 4 février 2016.

FILIP MISEV, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland • *E-mail :* `filip.misev@math.unibe.ch`

Mathematical subject classification (2010). — 57M25, 32S55; 20F55.

Key words and phrases. — Fibred knots and links, fibre surfaces, monodromy, cutting arcs, Hopf plumbing, singularities, Coxeter-Dynkin trees.

© Société Mathématique de France
doi:10.24033/bsmf.2747

arc of a plumbed Hopf band, and cutting along α amounts to deplumbing that Hopf band. In [4], Buck et al. give a simple criterion for when an arc preserves fibredness in terms of the monodromy $\varphi\colon S \to S$. As a corollary, they prove that each of the torus links of type $T(2,n)$ admits only a finite number of such arcs up to isotopy. It turns out that among torus links, this is an exception:

THEOREM 1. — *Let $n, m \geqslant 4$ or $n = 3, m \geqslant 6$. Then the fibre surface S of the torus link $T(n,m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

The remaining torus links $T(2,n)$, $T(3,3)$, $T(3,4)$ and $T(3,5)$ happen to be exactly those torus links that can also be obtained as plumbings of positive Hopf bands according to a finite tree, where vertices correspond to positive Hopf bands and edges indicate plumbing.

THEOREM 2. — *Let S be the fibre surface obtained by plumbing positive Hopf bands according to a finite tree T. There are, up to isotopy, only finitely many cutting arcs in S preserving fibredness, if and only if T is one of the Coxeter-Dynkin trees A_n, D_n, E_6, E_7 or E_8.*

To prove the "only if" part of Theorem 2, we consider orbits of a fixed arc under the monodromy to produce families of arcs that preserve fibredness. The basic idea is that such an orbit is infinite if the monodromy has infinite order. For example, we show that in fact every (prime) positive braid link with pseudo-Anosov monodromy admits infinitely many non-isotopic arcs preserving fibredness. This suggests the following question: is it true that among all (non-split prime) positive braid links, the ADE plane curve singularities are exactly those that admit just a finite number of fibredness preserving arcs up to isotopy?

Plan of the article. — We use the shorthand *ADE links* to refer to the links of the positive tree-like Hopf plumbings according to the trees A_n, D_n, E_6, E_7 or E_8. The subsequent section combines a criterion on arcs to preserve fibredness from [4] with the property of monodromies of positive Hopf plumbed surfaces to be right-veering. This allows for the following simple test for an arc to preserve fibredness, in our situation: an arc preserves fibredness if and only if it does not intersect its image under the monodromy (up to free isotopy).

Section 3 contains descriptions of the fibre surfaces and the monodromies of the links we consider (torus links and the ADE links). Alongside, we give a constructive proof of Theorem 1.

In Section 4, we explain the idea of proof for the finiteness result that provides the "if" part of Theorem 2, and list the fibred links obtained by cutting the fibre surfaces of the ADE links along an arc in Table 4.1.

Section 5 accounts for the cases where the monodromy has infinite order. This concerns in particular the positive tree-like Hopf plumbings that correspond to trees different from the ADE trees and settles the "only if" part of Theorem 2.

At the beginning of Section 6, we set up the notation and methods needed for the proof of the finiteness part of Theorem 2, which we split into Proposition 1 (concerning torus links) and Proposition 2 (concerning tree-like Hopf plumbings). The rest of that section is devoted to the proofs of these propositions.

Acknowledgements. — This work is part of the author's doctoral thesis under the guidance of Sebastian Baader. The author also wishes to thank the referee for helpful suggestions and comments.

2. Right-veering surface diffeomorphisms and cutting arcs that preserve fibredness

In the sequel we would like to make statements on the relative position of two arcs α, β in a surface S with boundary (that is, α, β are embedded intervals with endpoints on the boundary of S that are nowhere tangent to ∂S). The following definition will simplify matters.

Definition 1. — Let S be an oriented surface with boundary and let $\alpha, \beta \subset S$ be two arcs. A property $P(\alpha, \beta)$ is said to hold *after minimizing isotopies on α and β*, if $P(\tilde{\alpha}, \tilde{\beta})$ holds, where $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained from α, β by two isotopies (fixed at the endpoints) that minimize the geometric number of intersections between the two arcs.

The remainder of this section will recall the fact that every positive braid link (that is, the closure of a braid word consisting only of the positive generators of the braid group, without their inverses) is fibred and has so-called *right-veering* monodromy (see below for a definition). The torus links $T(n, m)$ provide examples, since they can be viewed as the closures of the positive braids $(\sigma_1 \cdots \sigma_{n-1})^m$, where the σ_i denote the (positive) standard generators of the braid group.

Definition 2 (see [8], Definition 2.1). — Let S be an oriented surface with boundary and $\varphi : S \to S$ a diffeomorphism that restricts to the identity on ∂S. Then φ is called *right-veering* if for every arc $\alpha : [0,1] \to S$, the vectors $((\varphi \circ \alpha)'(0), \alpha'(0))$ form an oriented basis after minimizing isotopies on α and $\varphi \circ \alpha$. This means basically that arcs starting at a boundary point of S get mapped "to the right" by φ.

It is known that every positive braid can be obtained as an iterated plumbing of positive Hopf bands (see [9]). Since a Hopf band is a fibre and plumbing