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NORM-INFLATION WITH INFINITE LOSS OF REGULARITY FOR PERIODIC NLS EQUATIONS IN NEGATIVE SOBOLEV SPACES

by Rémi Carles & Thomas Kappeler

Abstract. — In this paper we consider Schrödinger equations with nonlinearities of odd order $2\sigma + 1$ on $\mathbb{T}^d$. We prove that for $\sigma d \geqslant 2$, they are strongly illposed in the Sobolev space H^s for any $s < 0$, exhibiting norm-inflation with infinite loss of regularity. In the case of the one-dimensional cubic nonlinear Schrödinger equation and its renormalized version we prove such a result for H^s with $s < -2/3$.

Résumé (*Croissance de norme avec perte infinie de régularité pour les équations de Schrödinger périodiques en régularité négative*). — Nous considérons des équations de Schrödinger avec des non-linéarités d'ordre impair $2\sigma+1$ sur le tore $\mathbb{T}^d$. Nous montrons que pour $\sigma d \geqslant 2$, ces équations sont fortement mal posées dans l'espace de Sobolev H^s pour tout $s < 0$, avec en outre un phénomène de perte infinie de régularité. Dans le cas cubique mono-dimensionnel et sa version renormalisée, nous montrons le même résultat dans H^s, sous l'hypothèse $s < -2/3$.

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1. Introduction

We consider nonlinear Schrödinger (NLS) equations of the form

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \mu|\psi|^{2\sigma}\psi, \quad \psi = \psi(t,x) \in \mathbb{C},\ t \in \mathbb{R},\ x \in \mathbb{T}^d \tag{1.1}$$

and the renormalized versions

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \mu|\psi|^2\psi - \frac{2\mu}{(2\pi)^d}\left(\int_{\mathbb{T}^d}|\psi(t,x)|^2dx\right)\psi\,, \tag{1.2}$$

where $\sigma \geqslant 1$ is an integer, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\Delta = \sum_{k=1}^d \partial_{x_k}^2$, and $\mu \in \{1,-1\}$.

For any $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, denote by $\mathcal{F}L^{s,p}(\mathbb{T}^d) \equiv \mathcal{F}L^{s,p}(\mathbb{T}^d,\mathbb{C})$ the Fourier-Lebesgue space,

$$\mathcal{F}L^{s,p}(\mathbb{T}^d) = \{f \in \mathcal{D}'(\mathbb{T}^d,\mathbb{C}); \quad \langle\cdot\rangle^s\,\hat{f}(\cdot) \in \ell^p(\mathbb{Z}^d)\}$$

with $\ell^p(\mathbb{Z}^d) \equiv \ell^p(\mathbb{Z}^d,\mathbb{C})$ denoting the standard ℓ^p sequence space. Note that for any $s \in \mathbb{R}$, $\mathcal{F}L^{s,2}(\mathbb{T}^d)$ is the Sobolev space $H^s(\mathbb{T}^d) \equiv H^s(\mathbb{T}^d,\mathbb{C})$ and for any $1 \leqslant p \leqslant \infty$, $\cap_{s\in\mathbb{R}}\mathcal{F}L^{s,p}(\mathbb{T}^d)$ coincides with $C^\infty(\mathbb{T}^d) \equiv C^\infty(\mathbb{T}^d,\mathbb{C})$. The aim of this paper is to establish the following strong ill-posedness property of Equations (1.1) and (1.2).

Theorem 1.1. — *Let $\sigma, d \geqslant 1$ be integers.*

(i) *Assume that $d\sigma \geqslant 2$ in the case of* (1.1) *and $d \geqslant 2$ in the case of* (1.2). *Then for any $s < 0$, there exists a sequence of initial data $(\psi_n(0))_{n\geqslant 1}$ in $C^\infty(\mathbb{T}^d)$ such that*

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T}^d)} \underset{n\to\infty}{\longrightarrow} 0, \quad \forall p \in [1,\infty],$$

and a sequence of times $t_n \to 0$ such that the corresponding solutions ψ_n to (1.1) *respectively* (1.2) *satisfy*

$$\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T}^d)} \underset{n\to\infty}{\longrightarrow} \infty, \quad \forall r \in \mathbb{R},\ \forall p \in [1,\infty].$$

(ii) *If $d = \sigma = 1$, then for any $s < -2/3$, there exists a sequence of initial data $\psi_n(0) \in C^\infty(\mathbb{T})$ with*

$$\|\psi_n(0)\|_{\mathcal{F}L^{s,p}(\mathbb{T})} \underset{n\to\infty}{\longrightarrow} 0, \quad \forall p \in [1,\infty],$$

and a sequence of times $t_n \to 0$ such that the corresponding solutions ψ_n to (1.1) *respectively* (1.2) *satisfy*

$$\|\psi_n(t_n)\|_{\mathcal{F}L^{r,p}(\mathbb{T})} \underset{n\to\infty}{\longrightarrow} \infty, \quad \forall r \in \mathbb{R},\ \forall p \in [1,\infty].$$

Theorem 1.1 implies the following

Corollary 1.2. — *Let $d, \sigma \geqslant 1$ be integers and let s be as in Theorem 1.1. Furthermore assume that $p_1, p_2 \in [1,\infty]$ and $T > 0$. Then for no $r \in \mathbb{R}$, there exists a neighborhood U of 0 in $\mathcal{F}L^{s,p_1}(\mathbb{T}^d)$ and a continuous function*

$M_r : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that any smooth solution ψ to (1.1) *(or* (1.2)*) satisfy the a priori estimate*

$$\|\psi\|_{L^\infty(0,T;\mathcal{F}L^{r,p_2}(\mathbb{T}^d))} \leqslant M_r\left(\|\psi(0)\|_{\mathcal{F}L^{s,p_1}(\mathbb{T}^d)}\right).$$

In particular, for $p_1 = p_2 = 2$, there is no continuous function M_r such that smooth solutions to (1.1) *respectively* (1.2) *satisfy the a priori estimate*

$$\|\psi\|_{L^\infty(0,T;H^r(\mathbb{T}^d))} \leqslant M_r\left(\|\psi(0)\|_{H^s(\mathbb{T}^d)}\right).$$

Comments. — In connection with the study of ill-posedness of nonlinear Schrödinger and nonlinear wave equations on the whole space $\mathbb{R}^d$, Christ, Colliander, and Tao introduced in [10] (cf. also [11]), the notion of norm inflation with respect to a given (Sobolev) norm, saying that there exist a sequence of smooth initial data $(\psi_n(0))_{n\geqslant 1}$ and a sequence of times $(t_n)_{n\geqslant 1}$, both converging to 0, so that the corresponding smooth solutions ψ_n, evaluated at t_n, is unbounded. Further results in this direction were obtained in [2, 5, 6, 20], where in particular norm inflation together with finite or infinite loss of regularity was established for various equations on $\mathbb{R}^d$. Theorem 1.1 states that such type of results (in the strongest sense, since the loss of regularity is infinite) hold true for nonlinear Schrödinger equations on the torus $\mathbb{T}^d$.

Recently, the renormalized cubic Schrödinger Equation (1.2) has caught quite some attention. In particular, on $\mathbb{T}$, some well-posedness / ill-posedness results below L^2 have been established – see [9], [15] as well as [8], [18]. Although there are indications that (1.2) has better stability properties than (1.1), our results show no difference between the two equations as far as norm inflation concerns.

Finally let us remark that the scaling symmetry of (1.1), considered on the Sobolev spaces $H^s(\mathbb{R}^d)$, $\psi(t,x) \mapsto \lambda^{-2/\sigma}\psi(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ with $\lambda > 0$, has as critical exponent $s_{2,\sigma} = \frac{d}{2} - \frac{1}{\sigma}$ since for this value of s, the homogeneous H^s−norm is invariant under this scaling. More generally, for any given $1 \leqslant p \leqslant \infty$, the homogeneous $W^{s,p}(\mathbb{R}^d)$−norm is invariant for $s_{p,\sigma} = \frac{d}{p} - \frac{1}{\sigma}$. It suggests that the $\mathcal{F}L^{s,p}(\mathbb{R}^d)$−norm is invariant for $s^{FL}_{p,\sigma} = \frac{d}{p'} - \frac{1}{\sigma}$ with $\frac{1}{p'} = 1 - \frac{1}{p}$. Furthermore, the Galilean invariance of (1.1), $\psi(t,x) \mapsto e^{-iv\cdot x/2}e^{i|v|^2 t/4}\psi(t, x - vt)$ for arbitrary velocities v, leaves the $\mathcal{F}L^{0,p}(\mathbb{R}^d)$−norm invariant. Note that the statements of Theorem 1.1 for (1.1), considered on $H^s(\mathbb{T}^d)$, are valid in a range of s, contained in the half line $-\infty < s \leqslant \min(s_{2,\sigma}, 0)$.

Method of proof. — Let us give a brief outline of the proof of item (i) of Theorem 1.1 in the case of Equation (1.1). Following the approach, developed in [5] and [6] for equations such as nonlinear Schrödinger equations on the whole space $\mathbb{R}^d$, we introduce the following version of (1.1),

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad x \in \mathbb{T}^d$$