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***Abstract.* —** We investigate the Cauchy problem for the $f(R)$ theory of modified gravity, which is a generalization of Einstein's classical theory of gravitation. The integrand of the Einstein-Hilbert functional is the scalar curvature R of the spacetime, while, in modified gravity, it is a nonlinear function $f(R)$ so that, in turn, the field equations of the modified theory involve *up to fourth-order* derivatives of the unknown spacetime metric. We introduce here a *formulation of the initial value problem in modified gravity* when initial data are prescribed on a spacelike hypersurface. We establish that, in addition to the induced metric and second fundamental form (together with the initial matter content, if any), an initial data set for modified gravity must also provide one with the *spacetime scalar curvature* and its first-order time-derivative. We propose an *augmented conformal formulation* (as we call it), in which the spacetime scalar curvature is regarded as an *independent variable.* In particular, in the so-called wave gauge, we prove that the field equations of modified gravity are equivalent to a coupled system of *nonlinear wave-Klein-Gordon equations* with defocusing potential. We establish the consistency of the proposed formulation, whose main unknowns are the conformally-transformed metric and the scalar curvature (together with the matter fields) and we establish the existence of a maximal globally hyperbolic Cauchy development associated with any initial data set with sufficient Sobolev regularity when, for definiteness, the matter is represented by a massless scalar field. We analyze the so-called *Jordan coupling* and work with the so-called *Einstein metric*, which is conformally equivalent to the physical metric — the conformal factor depending upon the unknown scalar curvature. A main result in this paper is the derivation of quantitative estimates in suitably defined functional spaces, which are uniform in term of the nonlinearity $f(R)$ and show that spacetimes of modified gravity are *'close' to Einstein spacetimes*, when the defining function $f(R)$ is 'close' to the Einstein-Hilbert integrand R. We emphasize that this is a highly singular limit problem, since the field equations under consideration are fourth-order in the metric, while the Einstein equations are second-order only. In turn, our analysis provides the first mathematically rigorous validation of the theory of modified gravity.