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# SYMMETRY TYPES OF HYPERELLIPTIC RIEMANN SURFACES

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**Emilio Bujalance, Francisco-Javier Cirre, J.-M. Gamboa, Grzegorz Gromadzki**

***Abstract.* —** A compact Riemann surface $X$ is symmetric if it admits an antianalytic involution $\tau : X \to X$. Such an involution is called a real structure. Two real structures are isomorphic if they are conjugate in the full group $\mathrm{Aut}^{\pm}X$ of analytic and antianalytic automorphisms of $X$. In this memoir we classify up to isomorphism the real structures of all symmetric hyperelliptic Riemann surfaces of genus $g \geqslant 2$. The topological invariants of each isomorphism class are also computed. We give the list of groups which act as the full group of analytic and antianalytic automorphisms of such surfaces. Moreover, the complex algebraic curve associated to any such Riemann surface is described in terms of polynomial equations. We also find the explicit formula of a real structure in each isomorphism class.

***Résumé* (Types de symétrie des surfaces de Riemann hyperelliptiques)**

Une surface de Riemann compacte $X$ est dite symétrique si elle admet une involution antiholomorphe $\tau : X \to X$. On appelle structure réelle une telle involution. Deux structures réelles sont isomorphes si elles sont conjuguées par le groupe complet $\mathrm{Aut}^{\pm}X$ des automorphismes holomorphes et anti-holomorphes de $X$. Dans ce mémoire, nous classifions à isomorphisme près les structures réelles de toutes les surfaces de Riemann hyperelliptiques de genre $g \geqslant 2$. Nous calculons aussi les invariants topologiques de chaque classe d'isomorphisme. Nous donnons la liste des groupes qui agissent comme le groupe des automorphismes holomorphes et anti-holomorphes d'une telle surface. De plus, nous décrivons la courbe algébrique complexe associée à une telle surface en terme d'équations polynomiales. Nous donnons enfin une formule explicite pour une structure réelle dans chaque classe d'isomorphisme.

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