Revue d'histoire des mathématiques,
3 (1997), p. 1–48.

ALGEBRAIC GEOMETRY BETWEEN NOETHER AND NOETHER — A FORGOTTEN CHAPTER IN THE HISTORY OF ALGEBRAIC GEOMETRY

Jeremy GRAY (*)

ABSTRACT. — Mathematicians and historians generally regard the modern period in algebraic geometry as starting with the work of Kronecker and Hilbert. But the relevant papers by Hilbert are often regarded as reformulating invariant theory, a much more algebraic topic, while Kronecker has been presented as the doctrinaire exponent of finite, arithmetical mathematics. Attention is then focused on the Italian tradition, leaving the path to Emmy Noether obscure and forgotten.

There was, however, a steady flow of papers responding to the work of both Hilbert and Kronecker. The Hungarian mathematicians Gyula (Julius) König and József Kürschák, the French mathematicians Jules Molk and Jacques Hadamard, Emanuel Lasker and the English school teacher F.S. Macaulay all wrote extensively on the subject. This work is closely connected to a growing sophistication in the definitions of rings, fields and related concepts. The shifting emphases of their work shed light on how algebraic geometry owes much to both its distinguished founders, and how the balance was struck between algebra and geometry in the period immediately before Emmy Noether began her work.

RÉSUMÉ. — LA GÉOMÉTRIE ALGÉBRIQUE DE NOETHER À NOETHER — UN CHAPITRE OUBLIÉ DE L'HISTOIRE DE LA THÉORIE. — Mathématiciens et historiens considèrent en général que les travaux de Kronecker et de Hilbert inaugurent la période moderne de la géométrie algébrique. Mais on a souvent envisagé les articles correspondants de Hilbert comme une reformulation de la théorie des invariants, sujet de caractère nettement plus algébrique, alors que Kronecker était présenté comme promoteur doctrinaire d'une mathématique arithmétisée, finie. À partir de là, l'attention s'est portée sur la tradition italienne, laissant dans l'oubli la voie menant à Emmy Noether.

Et pourtant, il y eut un flux continu de publications, répondant aux travaux de Hilbert aussi bien que de Kronecker. Les mathématiciens hongrois Gyula (Julius) König et József Kürschák, les Français Jules Molk et Jacques Hadamard, Emmanuel

(*) Texte reçu le 17 mars 1997, révisé le 2 mai 1997.
Jeremy GRAY, Faculty of Mathematics and Computing, Open University, Milton Keynes, MK7 6AA, England. E-mail: j.j.gray@open.ac.uk.

Lasker et enfin le professeur de lycée anglais F.S. Macaulay, ont tous publié abondamment sur le sujet. Ces travaux sont étroitement liés à une élaboration progressive des notions d'anneau, de corps, et autres concepts connexes. L'évolution des préoccupations que manifestent ces publications fait ressortir de combien la géométrie algébrique est redevable à ses deux éminents fondateurs, et la façon dont se présentaient les rapports entre algèbre et géométrie dans la période immédiatement antérieure aux débuts de l'œuvre d'Emmy Noether.

INTRODUCTION

While there has been a considerable amount of historical work done on many topics in the history of mathematics around 1900, algebraic geometry continues to evade discussion, perhaps as befits the difficulty of the subject. It is difficult if not impossible to obtain an adequate treatment, of reasonable length and sophistication, of many of the key figures in the period and, as I hope to show here, many of the interesting and important minor figures have been completely forgotten.

The best literature (Dieudonné [1974], Shafarevich [1974]) rightly tells a story with Riemann as a vital influence and the theories of Riemann surfaces and Abelian functions as central topics. This soon divided into a transcendental enquiry and two algebraic-geometrical ones, one more algebraic, the other more geometrical. From the transcendental and the geometrical perspectives, Picard in France, Castelnuovo and Enriques in Italy are the respective dominant figures at the turn of the 20th century.[1] The algebraic-geometrical aspect was presented most notably by A. Brill and M. Noether, with extensions by such as Bertini. There was then an arithmetical theory developed by Hensel and Landsberg. What is strangely hard to find is accounts of a strand that flourished at the same time, and which is more visible today in many versions of what may be called classical algebraic geometry. In this area two major theorems are associated with David Hilbert: the basis theorem and the *Nullstellensatz* (or theorem of the zeros). For a history of these results one must turn to two classic papers: Hermann Weyl's obituary of Hilbert, and van der Waerden's notes on Hilbert's geometrical work, published in the 2nd volume of Hilbert's *Gesammelte Abhandlungen*. Dieudonné suggested, and the simplest scratching around confirms, that one of the major figures in the creation of an algebraic geometry of n dimensions was

[1] See Gray [1989] and Houzel [1991].

Leopold Kronecker, and he compared Kronecker's work with the different but overlapping theories of Dedekind and Weber.

It is hardly surprising that mathematicians had their way with the history of such a difficult subject for so long, although there is now a much more comprehensive account by Corry [1996]. Van der Waerden's three pages offer a classic account: Hilbert's many papers are reduced to two that really matter, the turning point in mathematicians' interests is neatly characterised (away from explicit formulae and towards conceptual clarification). One tradition ends, another gets off to a fine start with papers by Lasker, Macaulay and, in due course Emmy Noether and her school. Since these are indeed the origins of the ideas that dominated the subject for so long, the effect is that of a master telling you all you need to know. One realises that the past was surely messier, but is lulled into thinking that the details would make no significant difference. Weyl's account confirms this impression. It gives more details of the work in invariant theory, but ends with the same brief claim that on the foundations of Hilbert's work was erected the modern theory of polynomial ideals (for which we read commutative algebra).

Historians of algebraic geometry have taken their cue from the mathematicians. The subject of invariant theory is notoriously difficult, and one is understandably reluctant to contest a story that says that Hilbert put an end to it. The incentive is to treat the topic as background, part of the pre-history of algebraic geometry and the history of something else (group representation theory in the manner of Weyl, perhaps). It might seem odd that Hilbert's famous theorems arise in such an algebraic setting, but the whole relationship between commutative algebra and algebraic geometry is shrouded in just such ambiguities. Zariski and Samuel called their famous book *Commutative algebra* the child of an unborn parent. The parent, never to be written, was a book on algebraic geometry, which they called "*the main field of applications of, and the principal incentive for new research in, commutative algebra*" [1958, p. v]. The importance of commutative algebra is only underlined by the more avowedly geometrical treatise of Hodge and Pedoe, who introduced their third and final volume by invoking "*the needs of those geometers who are anxious to acquire the new and powerful tools provided by modern algebra, and who also want to see what they mean in terms of those ideas familiar to them*" [1954, p. vii].

It is not part of this paper to take the story up to the present day. But it should be noted that, if one examines later books on algebraic geometry, the most important novelties are surely the introduction of cohomology theories and, after Grothendieck, the language of schemes. In many ways Grothendieck's ideas produce the unification of commutative algebra and algebraic geometry that the mathematicians discussed in this paper seem to have regarded from afar.

The purpose of this paper is, rather, to explore the various historical problems that lie hidden behind the tidy histories and mathematical complexities. First, I look in more detail at the historical literature. Then we examine what Hilbert wrote, and then we consider Kronecker's contribution, notably his *Grundzüge* [1882], and try to see what it contained and what its influence was. It might seem that anyone who has radical opinions about the meaning of terms like $\sqrt{2}$, let alone π, would be hard to reconcile with a founding father of higher dimensional geometry. Indeed, most of Kronecker's contemporary geometers surely read the *Grundzüge*, if they read it at all, as if it referred to polynomials defined over the complex numbers. On the other hand, a modern mathematician feels that Kronecker's theory lacks the tools for dealing in depth with the problems of algebraic varieties. This raises questions about the response Kronecker's work could have elicited, and in pursuing them we shall find ourselves on a route that does indeed lead from Max to Emmy Noether.

1. SURVEY OF THE EXISTING HISTORICAL LITERATURE

There may not be a large historical literature, but it is still desirable not to regurgitate large amounts of it. I shall start therefore with Dieudonné's account of the two papers that Dedekind and Weber jointly and Kronecker published in 1882, and with the ideas about divisors that they contain. Dieudonné characterised these papers as opening up the whole analogy between algebraic geometry and algebraic number theory, and with introducing many ideas of abstract algebra that have become central but which in their day delayed reception of these works. As for Kronecker, Dieudonné argued [1974, pp. 60–61] that in his *Grundzüge* he gave precise definitions of the ideas of an irreducible variety and its dimension. (Dieudonné gave no precise reference, but the idea of dimension — *Stufe* — is defined in the *Grundzüge*, §10.) In order to give an intrinsic formulation of his ideas, Kronecker worked with ideals (which he called *Modulsysteme*) in polynomial rings; irreducible subvarieties give rise to prime ideals. In refining these ideas, Lasker [1905] obtained the primary decomposition theorem which became central in any discussion of the subject.

In Dieudonné's summary, the paper of Dedekind and Weber [1882] was directed to the algebraic theory of Riemann surfaces. They started from the field of functions associated to a Riemann surface, or, rather, from an algebraic extension of the field $\mathbf{C}(z)$ of rational functions in one variable. They introduced the concept of a discrete valuation (abstracting from the concrete notion of the zeros and poles of a function on a Riemann surface) and thus could associate a point set to the original field. Had they been able to topologise this set they would have been able to complete the circle and obtain a Riemann surface from a function field. But although they could not do that, they were able to show that finite sets of points, which they called polygons or divisors, and suitable equivalence classes of these, enabled one to recapture the Riemann-Roch theorem in this abstract setting. They did this by capturing at this abstract level the relevant properties of meromorphic differentials and of the canonical divisor, whence they could give a definition of the genus of the function field.

Dieudonné's account deals briskly with the first half of the paper where Dedekind and Weber drew out the analogy between number fields and function fields. Drawing on the work of their predecessors stretching back over fifty years, they defined an integer in a function field as an element ω which satisfies an equation of the form

$$\omega^e + b_1\omega^{e-1} + \cdots + b_{e-1}\omega + b_e = 0$$

where the coefficients $b_1, \dots, b_{e-1}, b_e$ are polynomials in z. The integral elements of a function field form a ring, which they denoted $\boldsymbol{o}$. Ideals in this ring are defined, and the standard operations on them introduced, including divisibility: the ideal $\boldsymbol{b}$ divides the ideal $\boldsymbol{a}$ if and only if $\boldsymbol{a}$ is a subset of $\boldsymbol{b}$. A prime ideal is one that is only divisible by itself and $\boldsymbol{o}$. Dedekind and Weber showed that every ideal is a product of prime ideals in a unique way, and that prime ideals correspond to points on the Riemann surface. At this point they commented, not for the first time, that the theory of divisibility was much simpler for number fields than function fields, and that in this matter the analogy broke down.