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NORMAL QUASI-ORDINARY SINGULARITIES

by

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Abstract. — We prove that any normal quasi-ordinary singularity is isomorphic to the normalization of a complete intersection that we get from the group of the quasi-ordinary projection. We give a new proof of the fact that any normal quasi-ordinary singularity is a germ of a toric variety. We also study some particular aspects of these singularities such as minimality, rationality and "cyclic quotient".

***Résumé* (Singularités quasi-ordinaires normales).** — Nous démontrons que toute singularité quasi-ordinaire normale est isomorphe à la normalisation d'une intersection complète que l'on détermine à partir du groupe de la projection quasi-ordinaire. Nous donnons une nouvelle preuve du fait qu'une singularité quasi-ordinaire normale est un germe de variété torique. Nous étudions certains aspects de ces singularités : rationalité, minimalité et « quotient cyclique ».

1. Introduction

An analytic germ of dimension n is quasi-ordinary when it is a local covering of $\mathbb{C}^n$, unramified outside the coordinate hyperplanes. These singularities became a subject of study with the so-called Jung's method that led to the first resolution of surface singularities.

They also appear as the "easiest" singularities. From different points of view they are a generalization of curve singularities. They can all be parameterized à la Puiseux ([**1**] and [**2**]). For hypersurfaces, J. Lipman exhibited from the Puiseux parameterization some *characteristic exponents* that determine the topological type of the embedded singularity ([**12**], see also [**10**]). For general quasi-ordinary normal singularities we refer to [**7**].

A full study of normal quasi-ordinary surfaces, linked with resolution of singularities can be found in [**3**, III.5]. A part of this work is dedicated to study generalizations of these results.

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We start by giving simple models for normal quasi-ordinary singularities: We prove that they are all normalization of some simple singularities that we determine from the group of the unramified covering they induce outside the coordinate hyperplanes. Then we link these models with toric varieties and prove that a normal quasi-ordinary singularity is a germ of an affine toric variety (see also [**14**, 2.3.4]).

As a corollary we prove that any local quasi-ordinary morphism of $\mathbb{C}^n$ is equivalent to a morphism of the form $(x_1, \dots, x_n) \mapsto (x_1^{a_1}, \dots, x_n^{a_n})$, for some positive integers $a_1, \dots, a_n$.

We study the case of finite cyclic quotient singularities, and give examples of normal quasi-ordinary singularities that are neither finite cyclic quotient nor minimal.

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2. The subgroup of a quasi-ordinary projection

Let $(X, 0)$ be a reduced and irreducible germ of analytic space of dimension n and let

$$(f, 0) : (X, 0) \longrightarrow (\mathbb{C}^n, 0)$$

be a germ of finite morphism (i.e. proper with finite fibers).

Given a representative $f : X \to U$ of the germ $(f, 0)$, there exists a nowhere dense subset B of U such that the restriction of f to $X \smallsetminus f^{-1}(B)$ is locally biholomorphic; in particular it is a topological covering of $U \smallsetminus B$ (see [**15**, 12.9]). The smallest analytic subset B of U with this property is called the *branching locus* of f. The map f is called an analytic covering.

Definition 2.1. — Let $(X, 0)$ be a germ of reduced and irreducible analytic space of dimension n. The germ $(X, 0)$ is quasi-ordinary if there exist a finite morphism $f : (X, 0) \to (\mathbb{C}^n, 0)$ and a local system of coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$ such that the branching locus of f is contained in the hypersurface of $\mathbb{C}^n$ defined by $x_1 \cdots x_n = 0$. Such a morphism is called a quasi-ordinary projection.

Let $(X, 0)$ be quasi-ordinary of dimension n and let $f : X \to U$ be a sufficiently small representative of a quasi-ordinary projection; U being a poly-disk around the origin in $\mathbb{C}^n$. Choose a system of coordinates $(x_1, \dots, x_n)$ in U, in such a way that the branching locus of f is contained in the space H defined by $x_1 \cdots x_n = 0$.

Set $U^* = U \smallsetminus H$ and $X^* = X \smallsetminus f^{-1}(H)$. The restricted map $f : X^* \to U^*$ is a topological covering. The space U^* is homeomorphic to the complex torus $\mathbb{C}^{*n}$. Since $\pi_1(U^*) \simeq \mathbb{Z}^n$ is abelian, the image of the induced map $f_* : \pi_1(X^*, x) \to \pi_1(U^*, u)$ does not depend on the choice of $x \in f^{-1}(u)$; we will call this image *the subgroup* of f and we will denote it by Γ_f.

We say that two analytic coverings $f : X \to U$ and $f' : X' \to U$ are equivalent if there exists an analytic isomorphism $h : X \to X'$ such that $f = f' \circ h$.

Proposition 2.2. — *Let $(X,0)$ and $(X',0)$ be normal quasi-ordinary germs. Two quasi-ordinary projections $f:(X,0)\to U$ and $f':(X',0)\to U$ are equivalent if and only if $\Gamma_f=\Gamma_{f'}$.*

Proof. — The topological coverings $f:X^*\to U^*$ and $f':X'^*\to U^*$ are equivalent if and only if $\Gamma_f=\Gamma_{f'}$ (see for example [**13**, th 6.6]). The isomorphism $X^*\simeq X'^*$ extends to $X\simeq X'$ by the Riemann extension theorem for normal complex spaces (see [**15**, 13.6]). □

3. Some simple quasi-ordinary singularities

Let $A:=(a_{i,j})_{1\leqslant i,j\leqslant n}$ be an invertible lower triangular matrix with non-negative integer entries and let m be a positive integer. Let $X_{A,m}$ be an irreducible component of the space defined in $\mathbb{C}^{2n}$ by the following equations in coordinates $(x_1,\dots,x_n,z_1,\dots,z_n)$:

$$
\begin{aligned}
z_1^m &= x_1^{a_{1,1}}\\
&\;\;\vdots\\
z_n^m &= x_1^{a_{n,1}}\cdots x_n^{a_{n,n}}
\end{aligned}
\tag{1}
$$

$X_{A,m}$ is of dimension n and contains the origin.

Consider the restriction to $X_{A,m}$ of the linear projection:

$$(x_1,\dots,x_n,z_1,\dots,z_n)\longmapsto(x_1,\dots,x_n)$$

and denote it by $f_{A,m}$.

The branching locus of the map $f_{A,m}$ is contained in the space defined by $x_1x_2\cdots x_n=0$. The space $X_{A,m}$ has then a quasi-ordinary singularity at the origin and $f_{A,m}$ is a quasi-ordinary projection.

We will now compute the subgroup of $f_{A,m}$.

Proposition 3.1. — *Let A be an invertible lower triangular $n\times n$-matrix with non-negative integer entries and let m be a positive integer. An n-tuple $b\in\mathbb{Z}^n$ is in the subgroup of the projection $f_{A,m}$ if and only if m divides all the entries of the vector Ab.*

Proof. — The canonical isomorphism $\varphi:\mathbb{Z}^n\to\pi_1(\mathbb{C}^{*n},(1,\dots,1))$ is given by $\varphi(b_1,\dots,b_n)(t)=(e^{b_1 2i\pi t},\dots,e^{b_n 2i\pi t})$.

The lifting of $\varphi(b_1,\dots,b_n)$ with base point $(1,\dots,1)$ is

$$L_{(b_1,\dots,b_n)}(t)=(e^{b_1 2i\pi t},\dots,e^{b_n 2i\pi t},e^{\frac{\sum_{j=1}^n a_{1,j}b_j}{m}2i\pi t},\dots,e^{\frac{\sum_{j=1}^n a_{n,j}b_j}{m}2i\pi t}).$$

It is a loop if and only if, for any $1\leqslant i\leqslant n$,

$$m \text{ divides } \sum_{j=1}^n a_{i,j}b_j.$$ □

Corollary 3.2. — *Let M be a lower triangular $n\times n$-matrix with integer entries. Suppose that the determinant of M is positive and that all the entries of the adjoint of M are non-negative so that $X_{\operatorname{Adj} M,\det M}$ is well defined. Then, the subgroup of the projection $f_{\operatorname{Adj} M,\det M}$ is the subgroup of $\mathbb{Z}^n$ generated by the vector columns of M.*

Proof. — An n-tuple $b=(b_1,\dots,b_n)\in\mathbb{Z}^n$ belongs to the subgroup of $\mathbb{Z}^n$ spanned by the vector columns of M if and only if there exists a vector $k\in\mathbb{Z}^n$ such that $b=Mk$. Since M is invertible

$$k=M^{-1}b=\frac{1}{\det M}(\operatorname{Adj} M)b$$

The right-hand side of the equality above has integer coordinates if and only if $\det M$ divides all the entries of the product $(\operatorname{Adj} M)b$. □

4. Characterization by the subgroups of $\mathbb{Z}^n$

We will now see that any subgroup of $\mathbb{Z}^n$ with finite index is the subgroup of a quasi-ordinary projection of type $f_{A,m}$.

Let Γ be a subgroup of $\mathbb{Z}^n$. There exists a system of generators $u_1,\dots,u_n$ of Γ such that $u_i=(0,\dots,0,u_{i,i},\dots,u_{n,i})$. We can get such a system by considering first a generator of $\Gamma\cap\{0\}\times\cdots\times\{0\}\times\mathbb{Z}$, call it u_n, then a generator of $\Gamma\cap\{0\}\times\cdots\times\{0\}\times\mathbb{Z}\times\mathbb{Z}$ and so on.

We will call such a system, a *lower triangular system of generators*. The matrix M, whose columns are the vectors $u_1,\dots,u_n$, is a lower triangular matrix.

Note that, by this process, the diagonal terms of M are unique up to a sign. If Γ is of finite index, then the diagonal terms are non-zero. The non-diagonal ones are determined up to a congruence modulo the diagonal term on their column ; therefore they can be chosen all non-positive.

Because of the choice of the entries of M and by linear calculus, all the entries of the adjoint matrix of M are non-negative integers.

Summarizing, we have:

Remark 4.1. — Let $\Gamma\subset\mathbb{Z}^n$ be a subgroup of finite index. There exists an invertible lower triangular matrix M such that, the adjoint of M has no negative entries and the vector columns of M generate Γ.

We can then define a space $X_{\operatorname{Adj} M,\det M}$ as in (1). By corollary 3.2, the subgroup of the canonical quasi-ordinary projection $f_{\operatorname{Adj} M,\det M}$ is precisely Γ.

Thus any subgroup of $\mathbb{Z}^n$ of finite index is the subgroup of a morphism of the type $f_{A,m}:X_{A,m}\to\mathbb{C}^n$. Moreover A can be chosen to be lower triangular and $m=\sqrt[n-1]{\det A}$.

Theorem 4.2. — *For any germ $(X,0)$ of normal quasi-ordinary singularity of dimension n there exists a lower triangular matrix A of order n and a positive integer m such* *that* $(X,0)$ *is isomorphic to the normalization of an irreducible space* $X_{A,m}$ *defined as in (1).*

Proof. — Let Γ be the subgroup of a quasi-ordinary projection associated to $(X,0)$. Let M be as in 4.1. By proposition 2.2, $(X,0)$ is isomorphic to the normalization of $(X_{\operatorname{Adj} M,\det M},0)$. □

Example 4.3. — Let Γ be the subgroup of $\mathbb{Z}^2$ generated by the lower triangular system $\{(1,-1),(0,2)\}$. Then any normal quasi-ordinary singularity of dimension 2 having Γ as subgroup for some quasi-ordinary projection is isomorphic to the normalization of an irreducible component of the space defined in $\mathbb{C}^4$ by:

$$\begin{aligned} z_1^2 &= x_1^2 \\ z_2^2 &= x_1 x_2. \end{aligned}$$

It is then isomorphic to the hypersurface of $\mathbb{C}^3$ defined by $z^2 = xy$.

Remark 4.4. — Theorem 4.2 generalizes the well known result for normal quasi-ordinary surfaces to normal quasi-ordinary singularities of any dimension and codimension (see [**3**, p. 82]).

5. Affine Toric varieties

In this section we will show that any normal quasi-ordinary singularity is a toric affine variety.

In [**10**], P. González Pérez proved theorem 5.2 stated below, for quasi-ordinary hypersurfaces of $\mathbb{C}^3$. In his Ph.D. thesis [**14**, 2.3.4], P. Popescu-Pampu gave an other proof for the same result, and as he says, his proof extends to general normal quasi-ordinary singularities. We give here a "hand-made" proof of that theorem.

Let Γ be a subgroup of $\mathbb{Z}^n$ of finite index. Let M be as in 4.1. If we call $v_1,\dots,v_n$ the rows of the matrix M^{-1}, then $(\det M)v_i$ is the i^{th} row of the adjoint matrix $\operatorname{Adj} M$.

Recall that $X_{\operatorname{Adj} M,\det M}$ is an irreducible component of the space defined by the ideal of $\mathbb{C}[X_1,\dots,X_n,Z_1,\dots Z_n]$ generated by $Z_i^{\det M} = X^{(\det M)v_i}$, $1 \leqslant i \leqslant n$; where $X^{(a_1,\dots,a_n)} = X_1^{a_1}\cdots X_n^{a_n}$.

Hence, the ring $\mathbb{C}[X_i, X^{v_j}, 1 \leqslant i,j \leqslant n]$ is isomorphic to the ring of regular functions of $X_{\operatorname{Adj} M,\det M}$. This leads us to speak about toric varieties.

We will introduce the main definitions and some properties of toric varieties that we will use. For more details and proofs we refer to [**9**].

Given a subgroup Γ of $\mathbb{Z}^n$, we call the dual of Γ and denote by Γ^* the group $\mathcal{H}om(\Gamma,\mathbb{Z})$. The intersection of Γ^* with the positive orthant σ_0 $(:= (\mathbb{R}_{\geqslant 0})^n)$ is a sub-semigroup of $\mathbb{Z}^n$.