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GENERAL ELEMENTS OF AN m-PRIMARY IDEAL ON A NORMAL SURFACE SINGULARITY

by

Romain Bondil

Abstract. — In this paper, we show how to apply a theorem by Lê D.T. and the author about linear families of curves on normal surface singularities to get new results in this area. The main concept used is a precise definition of *general elements* of an ideal in the local ring of the surface. We make explicit the connection between this notion and the more elementary notion of general element of a linear pencil, through the use of *integral closure of ideals*. This allows us to prove the invariance of the generic Milnor number (resp. of the multiplicity of the discriminant), between two pencils generating two ideals with the same integral closure (resp. the projections associated). We also show that our theorem, applied in two special cases, on the one hand completes, removing an unnecessary hypothesis, a theorem by J. Snoussi on the limits of tangent hyperplanes, and on the other hand gives an algebraic μ-constant theorem in linear families of planes curves.

***Résumé* (Éléments généraux d'un idéal m-primaire sur une singularité de surface normale)**
Dans ce travail, on expose des applications d'un théorème obtenu avec Lê D.T. sur les familles linéaires de courbes sur une singularité de surface normale. Le principal concept utilisé est une définition précise d'élements généraux dans un idéal m-primaire de l'anneau local de la surface. On explicite le lien qui existe entre cette notion et celle, plus élémentaire, d'élément général d'un pinceau linéaire grâce à la notion de clôture intégrale des ideaux.
Ceci permet de prouver l'invariance de la valeur du nombre de Milnor générique (resp. de la multiplicité du discriminant) si l'on considère différents pinceaux engendrant des idéaux de même clôture intégrale (resp. les projections associées).
Nous montrons aussi comment ce résultat complète, en enlevant une hypothèse inutile, un théorème de J. Snoussi sur les limites d'hyperplans tangents, et d'autre part donne aussi un théorème de type μ-constant algébrique pour les familles linéaires de courbes planes.

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Introduction

Let $(S,0)$ be a germ of normal complex-analytic surface, with local ring $\mathcal{O}_{S,0}$ corresponding to the germs of holomorphic functions on $(S,0)$, and maximal ideal m, formed by the germs taking the value 0 at 0.

To any couple (f,g) of elements of m, one may associate three related objects: the *linear pencil* of the curves $C_{\alpha,\beta} : \alpha f + \beta g = 0$ with $(\alpha,\beta) \in \mathbb{C}^2$, the *ideal* $J = (f,g)$ in $\mathcal{O}_{S,0}$, and the *projection*:

$$\begin{aligned} p : (S,0) &\longrightarrow (\mathbb{C}^2,0), \\ x &\longmapsto (f(x),g(x)). \end{aligned}$$

We will always assume that the curves $f=0$ and $g=0$ share no common component (in other words: the corresponding linear system has no *fixed component*, the ideal J is *m-primary*, and the projection p is *finite*).

Denoting by $(\Delta_p,0) \subset (\mathbb{C}^2,0)$ the discriminant of the projection p (see § 4), one may define a *general element of the pencil* $(C_{\alpha,\beta})$ as the inverse-image by p of any line $\alpha x + \beta y = 0$ in $\mathbb{C}^2$ which does not lie in the tangent cone of $(\Delta_p,0)$.

One may in turn define an element $h = af + bg \in J$ with $a,b \in \mathcal{O}_{S,0}$ to be *general* if, and only if, $a(0)f + b(0)g$ defines a general element of the pencil $(C_{\alpha,\beta})$.

In fact, we define here, for *any* m-primary ideal I in $\mathcal{O}_{S,0}$, a notion of *general element* which has the following property: take any pair (f,g) of elements of I such that the ideal $J=(f,g)$ is a *reduction* of I (see § 1), then the general elements of J (in the "pencil" sense) will be general elements of I, and conversely any general element of I will be obtained as an element of such a reduction.

However, this will not be our first definition of the *general elements of I* since we rather define them purely by their behaviour on the normalized blow-up of I (*cf.* def. 2.1).

In a previous paper, we proved that these elements are characterised by their Milnor number (theorem 2.3). Here, we focus on the applications of this result:

In § 3, we show how it covers both the study of limit of hyperplanes tangent to a normal surface, and the study of linear systems of plane curves, proving on one side a complement to a theorem by J. Snoussi, and on the other side an algebraic μ-constant theorem for linear systems of plane curves (also obtained by other means by E. Casas).

In § 4, we prove the relation between our definition of general elements of I and the one for pencils as claimed above. As a corollary, for two pencils (f,g) and (f',g') defining a reduction of I, the general elements of both pencils have the same Milnor number, and the discriminants of the corresponding projections have the same multiplicity.

1. Geometry of a theorem by Samuel

In this section only, we consider a germ $(X, 0)$ of complex analytic space with arbitrary dimension d. We let $\mathcal{O} := \mathcal{O}_{X,0}$ be the corresponding local analytic ring. In fact, the content of this section can be extended to any local noetherian ring with infinite residue field (see e.g. [**Li**] or [**Bo**] Chap. 2.3).

We recall that an element $f \in \mathcal{O}$ is said to be *integrally dependant* on an ideal I of $\mathcal{O}$ if it satisfies an equation:

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0,$$

with the condition $a_i \in I^i$ for all $i = 1, \ldots, n$.

The theory of integral dependance on ideals was initiated by O. Zariski (see [**S-Z**] Appendix 4) and under the influence of H. Hironaka was developed in the seminar [**LJ-Te**] where several characterisations are given. In the hands of B. Teissier, it became a cornerstone in the theory of equisingularity (see e.g. [**Te-2**] Chap. 1). More recently, the theory was extended to modules under the impulse of T. Gaffney (see the survey [**Ga-Ma**]).

Let us just mention that the set $\overline{I}$ of the elements of $\mathcal{O}$ integrally dependant on I is an ideal, called the integral closure of I in $\mathcal{O}$, and that the definition of integral closure finds a natural expression on the blow-up X_I of the germ $(X, 0)$ along I (see [**Te-2**]).

For the sake of simplicity, we restrict here to the case of a *reduced* germ $(X, 0)$ (*cf.* [**Bo**] *loc. cit.* for the general case). Then one may take the normalization $\overline{X_I}$ of the blow-up X_I, and following [**Te-2**] (Chap. 1, (1.3.6) et seq.), one proves that the equality $\overline{I} = \overline{J}$ of integral closures of ideals in $\mathcal{O}$ is equivalent to the equality:

$$I \cdot \mathcal{O}_{\overline{X_I}} = J \cdot \mathcal{O}_{\overline{X_I}}, \tag{1}$$

for the corresponding sheaves on the normalized blow-up $\overline{X_I}$.

We now take I to be an m-primary ideal of $\mathcal{O}$ *i.e.* containing a power m^s of the maximal ideal of $\mathcal{O}$.

Denoting by $\overline{b_I} : \overline{X_I} \to (X, 0)$ the normalized blow-up, we write $D_1, \ldots, D_s$ for the irreducible components of the reduced exceptional divisor $\mathcal{D} = |(\overline{b_I})^{-1}(0)|$, and v_{D_i} for the valuation along D_i.

Then we define (*cf.* [**B-L-1**] déf-prop. 1) an element $f \in I$ to be *v-superficial* if, and only if,

$$v_{D_i}(f) = v_{D_i}(I) := \inf\{v_{D_i}(g), g \in I\} \text{ for all } i = 1, \ldots, s. \tag{2}$$

Denoting by $D_f := \sum_{i=1}^{s} v_{D_i}(f) D_i$, the total transform $(f)^* := (f \circ \overline{b_I})$ on $\overline{X_I}$ may be written as a sum of divisors:

$$(f)^* = (f)' + D_f,$$

with $(f)'$ the strict transform of f on $\overline{X_I}$.

The first part of the following proposition is an avatar of a theorem by P. Samuel. The second part is the geometric version announced in the title:

Proposition 1.1

i) *Let $\mathcal{O}$ be a local noetherian ring of dimension d with infinite residue field $\mathcal{O}/m$. Let I be an m-primary ideal of $\mathcal{O}$. There exists a d-tuple $(f_1, \dots, f_d)$ of elements of I such that the ideal $(f_1, \dots, f_d)$ is* a reduction *of I, i.e. has the same integral closure as I.*

ii) *In our setting, let $\mathcal{O}$ be the local ring of a reduced analytic germ $(X, 0)$. The d-tuples in i) are characterized by the two conditions that first, all the f_i are v-superficial in I and secondly, the intersection of their strict transforms $(f_i)'$ with the exceptional divisor $\mathcal{D}$ on the normal blow up of I verifies:*

$$(f_1)' \cap (f_2)' \cap \cdots \cap (f_d)' \cap \mathcal{D} = \varnothing.$$

We call such a d-tuple a good *d-tuple of v-superficial elements in I.*

We will not give the proof here, but the reader should understand that ii) also easily gives the proof of i) thanks to the characterisation on (1) above. In fact, the same "geometric proof" works under the general hypotheses of i) but one has to work on the non normalized blow-up (see [**Bo**] Chap. 2).

The original theorem by Samuel was formulated in terms of multiplicities (*cf.* [**S-Z**] Chap. VIII thm. 22) so that it seems relevant to mention the following:

Proposition 1.2. — *Let $\mathcal{O}$ be analytic local integral domain, and I an m-primary ideal of $\mathcal{O}$. The multiplicity $e(I/(f), \mathcal{O}/(f)) = e(I, \mathcal{O})$ if, and only if, f is v-superficial.*

This result can be deduced from a general formula for $e(I/(f), \mathcal{O}/(f))$ due to Flenner and Vogel in [**Fl-Vo**] (for any noetherian local ring).

2. General elements of an ideal

From now on, we restrict ourselves to a two-dimensional *normal* germ $(S, 0)$.(1)

Definition 2.1. — Let $\mathcal{O}$ be the local ring of a germ of normal surface $(S, 0)$ and let I be an m-primary ideal of $\mathcal{O}$. Adapting the notation from section 1, $\overline{S_I}$ denotes the normalized blow-up of I on $(S, 0)$. We define an element $f \in I$ to be *general* if, and only if,

(i) f is v-superficial in I (*cf.* § 1 (2)),

(ii) the strict transform $(f)'$ is a smooth curve transversal to the exceptional divisor $\mathcal{D}$ in $\overline{S_I}$, which means that $(f)'$ does not go through singular points either of $\overline{S_I}$ or of $\mathcal{D}$ and that the intersection is transverse.

(1)For the elementary properties of normal surfaces we use here, see [**Sn**] § 2.6, and [**B-L-2**].

Consider any resolution $r : X \to \overline{S_I}$ of the singularities of $\overline{S_I}$, *good* in the sense that, denoting by $\pi = \overline{b_I} \circ r : X \to (S,0)$, the exceptional divisor $Z = \pi^{-1}(0)$ has only normal-crossing singularities.

Denote by $(f \circ \pi) = (f)' + Z_f$ the decomposition of the total transform of (f) on X into an exceptional (compact) part Z_f and its strict transform denoted again $(f)'$.

Denoting Z_I the divisor defined by $I \cdot \mathcal{O}_X$ on X, we easily get the following:

Proposition 2.2. — *With the notation as above, $f \in I$ is general if, and only if, its total transform on X is such that:*

α) *its exceptional part is the generic one for the elements of I i.e. $Z_f = Z_I$,*

β) *its strict transform is a (multi-germ of) smooth curves transversal to Z.*

As a corollary to this proposition, it is possible (either by a computation of Euler-Poincaré characteristic of covering spaces as indicated in [**B-L-1**] § 4, which followed [**GS**], or by an algebraic derivation from a Riemann-Roch formula as in [**Mo**] 2.1.4) to compute the Milnor number (in the sense of [**Bu-Gr**]) of the complex curve defined by any general element $f \in I$. We then get:

$$\mu(f) = \mu_I := 1 - (Z_I.(Z_I - |Z_I| - K)), \tag{3}$$

on any good resolution as defined before the proposition, where $|Z_I|$ (resp. K) denote the reduced divisor associated to Z_I (resp. the numerically canonical cycle) and $(\,\cdot\,)$ denotes the intersection product (see [**B-L-1**] or [**Bo**] chap. 3 for more details).

The main theorem in [**B-L-1**] is the converse implication:

Theorem 2.3. — *Let $(S,0)$ a germ of normal surface singularity, and I an m-primary ideal of $\mathcal{O}_{S,0}$. An element $f \in I$ is general in the sense of 2.1 if, and only if, the Milnor number $\mu(f)$ has the value μ_I prescribed by formula (3), which is also the minimum Milnor number for the elements of I.*

Remark 2.4. — Thanks to the algebraic computation of the Milnor number for general elements which follows from [**Mo**] (see before formula (3)), theorem 2.3 is proved without any topological argument, so that the proof fits to the setting of algebraic geometry over any algebraically closed field of characteristic zero.

3. Two special cases

3.1. The case when $(S,0)$ is arbitrary but $I = m$. — For a germ $(S,0)$ of normal surface, given an embedding $(S,0) \subset (\mathbb{C}^N,0)$ defined by N generators of the maximal ideal m of $\mathcal{O}_{S,0}$, we may consider the elements $f \in m$ as hypersurface sections of S. From this point of view, J. Snoussi studies in [**Sn**] what he calls the *general hyperplanes* with respect to $(S,0)$. An hyperplane $H \ni 0$ in $\mathbb{C}^N$ is said to be *general* for $(S,0)$ if, and only if, it is not the limit of hyperplanes tangent to the non