DETERMINATION OF LIPSCHITZ STRATIFICATIONS
FOR THE SURFACES $y^a = z^b x^c + x^d$

by

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Abstract. — We determine Lipschitz stratifications for the family of surfaces $y^a = z^b x^c + x^d$, where a, b, c, d are positive integers.

***Résumé* (Détermination de stratifications de Lipschitz pour les surfaces $y^a = z^b x^c + x^d$)**
   Nous déterminons des stratifications de Lipschitz pour la famille de surfaces $y^a = z^b x^c + x^d$, où a, b, c, d sont des entiers positifs.

1. Introduction and previous results

R. Thom and H. Whitney suggested the use of stratifications as a method of understanding the geometric structure of singular analytic spaces. The stronger the conditions imposed on the stratified set, the better the understanding of the geometry of this set. One of the strongest conditions is the L-regularity introduced by T. Mostowski in 1985 [**3**]. Mostowski introduced the notion of Lipschitz stratification and proved its existence for complex analytic sets. The existence of Lipschitz stratification for subanalytic sets and real analytic sets was later proved by A. Parusinski [**5**], [**6**], [**7**]. The Lipschitz stratifications ensure bi-Lipschitz triviality of the stratified set along each stratum, and bi-Lipschitz homeomorphisms preserve sets of measure zero, order of contact, and Lojasiewicz exponents. The L-conditions are preserved after intersection with generic wings, that is L-regularity implies L^*-regularity [**2**]; this was one of the conditions required of a good equisingularity notion by B. Teissier in his foundational 1974 Arcata paper [**8**].

In this paper, we check Mostowski's conditions for almost all members ($\sim 99\%$) of the family of surfaces with two strata given by $(V - Oz, Oz)$ where V is the algebraic surface

$$\{(x, y, z) \mid y^a = z^b x^c + x^d\}$$

in $\mathbb{R}^3$ or $\mathbb{C}^3$, with a, b, c, d positive integers.

The character C in this paper will stand for various constants.

We begin by recalling the definition of Lipschitz stratification due to Mostowski [**3**].

Let X be a closed subanalytic subset of an open subset of $\mathbb{R}^n$. By a *stratification* of X we shall mean a family $\Sigma = \{S^j\}_{j=l}^m$ of closed subanalytic subsets of X defining a filtration:
$$X = S^m \supset S^{m-1} \supset \cdots \supset S^l \neq \varnothing$$
and $\overset{\circ}{S^j} = S^j \smallsetminus S^{j-1}$, for $j = l, l+1, \ldots, m$ (where $S^{l-1} = \varnothing$), is a smooth manifold of pure dimension j or empty. We call the connected components of $\overset{\circ}{S^j}$ the *strata* of Σ. We denote the function measuring distance to S^j by d_j, so that $d_j(q) = \mathrm{dist}(q, S^j)$. Set $d_{l-1} \equiv 1$, by convention (this will be used in the definitions below).

Definition 1.1. — Let $\gamma > 1$ be a fixed constant. A *chain* for a point $q \in \overset{\circ}{S^j}$ is a (strictly) decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = l$ such that each j_s, $s \geqslant 2$, is the greatest integer less than j_{s-1} for which
$$d_{j_s-1}(q) \geqslant 2\gamma^2 d_{j_s}(q).$$

For each $j_s \in \{j_1, \ldots, j_r\}$ choose a point $q_{j_s} \in \overset{\circ}{S^{j-s}}$ such that $q_{j_1} = q$ and $|q - q_{j_s}| \leqslant \gamma d_{j_s}(q)$.

If there is no confusion, we will call the sequence of points (q_{j_s}) a chain of q.

For $q \in \overset{\circ}{S^j}$, let $P_q : \mathbb{R}^n \to T_q \overset{\circ}{S^j}$ be the orthogonal projection to the tangent space $T_q \overset{\circ}{S^j}$ and let $P_q^\perp = I - P_q$ be the orthogonal projection onto the normal space $(T_q \overset{\circ}{S^j})^\perp$.

Definition 1.2. — A stratification $\Sigma = \{S^j\}_{j=l}^m$ of X is said to be a Lipschitz stratification (or to satisfy the L-conditions) if for some constant $C > 0$ and for every chain $q = q_{j_1}, q_{j_2}, \ldots, q_{j_r}$ and every k, $2 \leqslant k \leqslant r$,

$(L1)$ $$|P_q^\perp P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q - q_{j_2}|/d_{j_k-1}(q)$$

and for each $q' \in \overset{\circ}{S^{j-1}}$ such that $|q - q'| \leqslant (1/2\gamma)d_{j_1-1}(q)$,

$(L2)$ $$|(P_q - P_{q'})P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q - q'|/d_{j_k-1}(q)$$

and

$(L3).$ $$|P_q - P_{q'}| \leqslant C|q - q'|/d_{j_1-1}(q)$$

2. Classifications and Calculations

In this section we give diagrams showing when $L-$regularity holds for the stratification with two strata given by $(V - Oz, Oz)$ where V is the germ of the algebraic surface
$$\{(x, y, z) \mid y^a = z^b x^c + x^d\}$$
in $\mathbb{R}^3$ or $\mathbb{C}^3$, and a, b, c, d are positive integers.

Diagrams classifying Whitney a-regularity and b-regularity for this family of algebraic surfaces were obtained by the second author ([**9**] and [**10**]), while the finer classification needed for the Kuo-Verdier w-regularity can be found in the thesis of L. Noirel [**4**].

Calculations. — It is easy to see that the $(L1)$ condition in the definition of Lipschitz stratification of Mostowski implies the Kuo-Verdier (w) condition, and in our case (where there are only two strata) is actually equivalent to (w). So we only need to study cases when (w) is satisfied in the classification obtained by L. Noirel. And for these we only need to check condition $(L2)$ (for $k = 2$) and $(L3)$. Here we illustrate the method of obtaining the classification with details of the calculations deciding several branches of Diagram 2 (see the end of the paper). The essential technique in checking $(L2)$ and $(L3)$ is to apply the mean value theorem to compare values at the two points q and q' whose distance apart is controlled.

Complete detailed calculations for the case of Lipschitz stratifications can be found in the first author's thesis [**1**].

(2.1) If $a = 1$, V is the graph of a smooth function, hence V is a smooth submanifold of $\mathbb{R}^3$, and consequently is (L) regular.

The Lipschitz condition of Mostowski is stronger than (b)-regularity and (w)-regularity, so using the classifications of Trotman in [**9**] and Noirel in [**4**] we have:

(2.2) If $a \neq 1$, $d > c$, and b is odd, the conditions (b) and (w) do not hold, and hence the stratification is not (L) regular.

(2.3) If $a \neq 1$, $d > c$, b is even and $(d - c)$ is odd, the conditions (b) and (w) do not hold, and hence the stratification is not (L) regular.

(2.4) If $a \neq 1$, $d > c$, b is even, $(d-c)$ is even and $a(d-c) > b|d-a|$, the condition (w) does not hold, and hence the stratification is not (L) regular.

(2.5) If $a \neq 1$, $d > c$, b is even, $(d - c)$ is even, $a(d - c) \leqslant b|d - a|$, and $c < a \leqslant d$, the conditions (b) and (w) do not hold, and hence the stratification is not (L) regular.

Notation. — Let f and g be two real valued functions defined on the same set. We will write $f \ll g$ if there exists a function α which tends to 0 at 0, such that $f \leqslant \alpha g$. We also will write $f \approx g$ if there is a positive constant C such that $f \leqslant Cg$ and $g \leqslant Cf$. For functions defined on $\mathbb{R}^3$, we will use this notation for their restrictions to V.

Proposition 2.1. — *Consider the stratification with two strata $(V - Oz, Oz)$ of the germ $V = \{(x, y, z) \mid y^a = z^b x^c + x^d\}$ in $\mathbb{R}^3$, where a, b, c, d are positive integers. Suppose $c \geqslant d$. Then the stratification is Lipschitz.*

Proof. — Let $V = \{(x, y, z) \mid f(x, y, z) = -y^a + z^b x^c + x^d\}$, so that $\operatorname{Sing} V = Oz$.

Let (q, q_{j_2}) be a chain, with $q = (x, y, z) \in \operatorname{Reg} V$ and $q_{j_2} \in O_z$. We shall write $\partial_q f$ for the gradient of f at q from now on.

$$\text{Then} \quad \partial_q f = (cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1}).$$

Then $d_{j_2-1}(q) = 1$, and $d_{j_1-1}(q) = d(q, O_z)$. By the definition of orthogonal projection, we have the following inequalities:

$$|P_q - P_{q'}| = |P_q^\perp - P_{q'}^\perp|, \quad \text{and} \quad P_q^\perp(v) = \left\langle v, \frac{\partial_q f}{\|\partial_q f\|} \right\rangle \frac{\partial_q f}{\|\partial_q f\|}.$$

Suppose $c \geqslant d$. In this case (w)-regularity holds, so that to verify that the stratification is Lipschitz it is enough to check $(L2)$ and $(L3)$ in the definition of Lipschitz stratification, because for a stratification composed of two strata, (w) is equivalent to $(L1)$.

We work in a neighbourhood of the origin. By definition of V, we have $y = \pm x^{d/a}(z^b x^{c-d} + 1)^{1/a}$ and we see that $y \approx x^{s\frac{d}{a}}$.

I. Case $a < d$. — We have that $y \approx x^{d/a}$ and $a < d$, hence $|y| = \circ|x|$ and $d(q, O_z) \approx |x|$. Take $q' = (x', y', z') \in V - Oz$, such that $|q - q'| \leqslant \frac{1}{2} d(q, O_z)$. That is:

$$|q - q'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \leqslant \frac{1}{2}|x|.$$

It is easy to see that $xx' > 0$. We shall treat separately the cases $yy' < 0$ and $yy' \geqslant 0$.

If $yy' < 0$, then as $xx' > 0$, it follows that $|y| + |y'| \leqslant C|q - q'|$. For a point $q = (x, y, z)$, we have:

$$\left|\frac{\partial f}{\partial x}\right| = |cx^{c-1}z^b + dx^{d-1}| \approx |x^{d-1}| \quad \text{because } c \geqslant d$$
$$\left|\frac{\partial f}{\partial y}\right| = |ay^{a-1}| \approx |x^{d-d/a}|$$
$$\left|\frac{\partial f}{\partial z}\right| = |bx^c z^{b-1}|.$$

As $a < d$, in some neighbourhood of 0 we have:

$$\left|\frac{\partial f}{\partial x}\right| \ll \left|\frac{\partial f}{\partial y}\right| \quad \text{and} \quad \left|\frac{\partial f}{\partial z}\right| \ll \left|\frac{\partial f}{\partial y}\right|.$$

Calculating,

$$\left|\frac{bx^c z^{b-1}\partial_q f}{\|\partial_q f\|^2}\right| = \left|\frac{bx^c z^{b-1}(cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1})}{(cx^{c-1}z^b + dx^{d-1})^2 + (-ay^{a-1})^2 + (bx^c z^{b-1})^2}\right|$$
$$\approx \left|\frac{x^c z^{b-1}}{y^{a-1}}\right|$$

since $\left|\partial f/\partial x\right| \ll \left|\partial f/\partial y\right|$ and $\left|\partial f/\partial z\right| \ll \left|\partial f/\partial y\right|$ in some neighbourhood of 0, and

$$\left|\frac{x^c z^{b-1}}{y^{a-1}y}\right| = \left|\frac{x^c z^{b-1}}{z^b x^c + x^d}\right| < \left|\frac{x^c z^{b-1}}{x^d}\right|$$
$$\approx \left|x^{c-d} z^{b-1}\right| < 1 \quad \text{in some neighbourhood of 0.}$$

Then

$$\left|(P_q - P_{q'})P_{q_{j_2}}\right| = \left|(P_q - P_{q'})(0,0,1)\right| = \left|(P_q^\perp - P_{q'}^\perp)(0,0,1)\right|$$
$$= \left|\frac{bx^c z^{b-1}\partial_q f}{\|\partial_q f\|^2} - \frac{bx'^c z'^{b-1}\partial_{q'} f}{\|\partial_{q'} f\|^2}\right|$$
$$\leqslant |y| + |y'| < C|q - q'|.$$

We have shown that $(L2)$ holds when $yy' < 0$.

Now we check $(L3)$ when $yy' < 0$. Take the basis vectors: $(1,0,0), (0,1,0)$ and $(0,0,1)$. Calculating:

$$\left|\frac{(cx^{c-1}z^b + dx^{d-1})\partial_q f}{\|\partial_q f\|^2}\right| = \left|\frac{(cx^{c-1}z^b + dx^{d-1})(cx^{c-1}z^b + dx^{d-1}, -ay^{a-1}, bx^c z^{b-1})}{(cx^{c-1}z^b + dx^{d-1})^2 + (-ay^{a-1})^2 + (bx^c z^{b-1})^2}\right|$$
$$\approx \left|\frac{cx^{c-1}z^b + dx^{d-1}}{y^{a-1}}\right|$$

since $\left|\partial f/\partial x\right| \ll \left|\partial f/\partial y\right|$ and $\left|\partial f/\partial z\right| \ll \left|\partial f/\partial y\right|$ in some neighbourhood of 0, and

$$\left|\frac{cx^{c-1}z^b + dx^{d-1}}{ay^{a-1}}\right|\frac{|x|}{|y|} \text{ is bounded in some neighbourhood of 0.}$$

Hence

$$\left|(P_q - P_{q'})(1,0,0)\right| = \left|(P_q^\perp - P_{q'}^\perp)(1,0,0)\right|$$
(i)
$$= \left|\frac{(cx^{c-1}z^b + dx^{d-1})\partial_q f}{\|\partial_q f\|^2} - \frac{(cx'^{c-1}z'^b + dx'^{d-1})\partial_{q'} f}{\|\partial_{q'} f\|^2}\right|$$
$$\leqslant \left|\frac{y}{x}\right| + \left|\frac{y'}{x'}\right| \leqslant C\frac{|q - q'|}{d(q, O_z)}.$$

We do the same for the basis vector $(0,1,0)$. Because $\left|\partial f/\partial x\right| \ll \left|\partial f/\partial y\right|$ and $\left|\partial f/\partial z\right| \ll \left|\partial f/\partial y\right|$ in a neighbourhood of 0,

$$\left|\frac{ay^{a-1}\partial_q f}{\|\partial_q f\|^2}\right| \quad \approx \quad \left|\frac{(y^{a-1})^2}{\|\partial_q f\|^2}\right|.$$

Also

$$|(P_q - P_{q'})(0,1,0)| = |(P_q^\perp - P_{q'}^\perp)(0,1,0)|$$
$$= \left|\frac{ay^{a-1}\partial_q f}{\|\partial_q f\|^2} - \frac{ay'^{a-1}\partial_{q'} f}{\|\partial_{q'} f\|^2}\right|.$$