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ARC-ANALYTICITY IS AN OPEN PROPERTY

by

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Abstract. — We prove that the locus of the points where a bounded continuous subanalytic function is not arc-analytic, is a closed nowhere dense subanalytic set. This shows that the property of being arc-analytic at a point, is an open property.

***Résumé* (L'arc-analyticité est une propriété ouverte). —** Nous montrons que l'ensemble de points où une fonction sous-analytique, bornée et continue n'est pas arc-analytique est un ensemble sous-analytique fermé. Autrement dit : la propriété d'être arc-analytique en un point est une propriété ouverte.

1. Introduction

Let U be an open subset of $\mathbb{R}^n$. Following [**9**] we say that a map $f : U \to \mathbb{R}^k$ is *arc-analytic* if for any analytic arc $\alpha : (-\varepsilon, \varepsilon) \to U$, $f \circ \alpha$ is also analytic. In general arc-analytic maps are very far from being analytic, in particular there are arc-analytic functions which are not subanalytic [**11**], not continuous [**3**], with a non-discrete singular set [**12**]. Hence it is natural to consider only arc-analytic maps with subanalytic graphs. Earlier T.-C. Kuo, motivated by equisingularity problems, introduced in [**8**] the notion of *blow-analytic* functions, *i.e.*, functions which become analytic after a composition with appropriate proper bimeromorphic maps (e.g. a composition of blowings up with smooth centers). Clearly any blow-analytic mapping is arc-analytic and subanalytic. The converse holds in a slightly weaker form [**2**] (see also [**16**]). Blow-analytic maps have been studied by several authors (see the survey [**4**]). It is known that in general subanalytic and arc-analytic functions are continuous [**9**], but not necessarily (locally) Lipschitz [**4**], [**17**].

The main result of this note is Theorem 3.1, which claims that the locus of the points at which a bounded, continuous, subanalytic function $f : U \to \mathbb{R}$ is not arc-analytic, is a closed subanalytic subset of U. In other words, if f is analytic on any germ of analytic arc at a given point $a \in U$, then f is arc-analytic in a neighbourhood of a.

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This property is of interest if we deal with germs of arc-analytic functions. For instance let us recall the main result of [**13**]. It states the following: if g is an arc-analytic function, such that for some natural r the function $f = g^r$ is analytic, then g is locally Lipschitz. Moreover, if r is less than the multiplicity of f, then g is C^1. Now, if we are interested in a local version of this result, thanks to our Theorem 3.1, it is enough to check the arc-analyticity of g only on analytic arcs passing through a given point.

The main tool in the proof of Theorem 3.1 is Parusiński's *Rectilinearization of subanalytic function* [**16**]. We thank the referee for careful reading and valuable remarks.

2. Definitions – Notations

2.1. Locally blow-analytic functions. — We recall some of the notions used in this paper (for more information see for instance [**3**], [**4**], [**5**], [**8**], [**11**], [**12**], [**18**]).

We recall first a definition of a local blowing up. Let M be an analytic manifold and $\Omega \subset M$ an open set. Assume that X is an analytic submanifold of M, closed in Ω. Then we can define the mapping $\tau : \widetilde{\Omega} \to \Omega$, the blowing up of Ω with the centre X, see for instance [**7**] or [**14**]. A restriction of τ to an open subset of $\widetilde{\Omega}$ is called *a local blowing up with a smooth (nowhere dense) centre*. Local blowings up have the important arc lifting property. We state it precisely below:

***Lemma 2.1* (Arc lifting property).** — *Let M be an analytic manifold and let $\sigma : W \to M$ be a finite composition of local blowings up with smooth centres. Assume that $\gamma : (-\varepsilon, \varepsilon) \to M$ is an analytic arc, $\gamma((-\varepsilon,\varepsilon)) \subset \sigma(W)$. Then there exists an analytic arc $\widetilde{\gamma} : (-\varepsilon,\varepsilon) \to W$ such that $\sigma \circ \widetilde{\gamma} = \gamma$.*

Let U be a neighbourhood of the origin of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ denote a map defined on U except possibly some thin subset of U. We say that f is *locally blow-analytic* via a locally finite collection of analytic modifications $\sigma_\alpha : W_\alpha \to \mathbb{R}^n$, if for each α we have

i) W_α is isomorphic to $\mathbb{R}^n$ and σ_α is the composition of finitely many local blowings up with smooth nowhere dense centres, and $f \circ \sigma_\alpha$ has an analytic extension on W_α.

ii) There are subanalytic compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup \sigma_\alpha(K_\alpha)$ is a neighbourhood of $\overline{U}$.

The notion of *(locally) blow-analytic* functions (or maps) is very much related to the notion of *arc-analytic* functions, *i.e.*, functions $f : U \to \mathbb{R}$ such that $f \circ \alpha$ is analytic for any analytic arc $\alpha : I \to U$, here U is an open subset of $\mathbb{R}^n$ and I is an open interval. Indeed in [**2**], see also [**16**], it is proved that an *arc-analytic* function has *subanalytic* graph if and only if it is *locally blow-analytic*.

Let $f : U \to \mathbb{R}$ be a subanalytic function defined in an open subset of $\mathbb{R}^n$. We will say that f *is not arc-analytic at a point* $x \in U$, if there exists an analytic arc $\gamma : (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = x$ and the composed function $f \circ \gamma$ is not analytic at $t = 0$.

3. Main Results

Our main result is the following theorem.

Theorem 3.1. — *Let $f : U \to \mathbb{R}$ be a bounded continuous subanalytic function defined in an open subset of $\mathbb{R}^n$. Then the locus of the points in U at which f is not arc-analytic, is a closed, nowhere dense, subanalytic subset of U.*

Remark 3.2. — If f is semialgebraic, then the locus of the points in U at which f is not arc-analytic, is a closed, nowhere dense, semialgebraic subset of U.

Proof. — Let us denote

$$S_{\mathrm{naa}}(f) = \{x \in U \mid f \text{ is not arc-analytic at } x\}.$$

Clearly the set $S_{\mathrm{naa}}(f)$ is contained in the singular set of f:

$$S_{\mathrm{na}}(f) = \{x \in U \mid f \text{ is not analytic at } x\}.$$

It is known ([**19**], [**10**], [**1**]), that the set $S_{\mathrm{na}}(f)$ is subanalytic, closed and nowhere dense in U (*i.e.*, $\dim S_{\mathrm{na}}(f) \leqslant n-1$). However, in general, the set $S_{\mathrm{na}}(f)$ is larger than the set $S_{\mathrm{naa}}(f)$. Our proof follows an idea from [**10**] and it uses some facts on subanalytic functions of one variable.

Lemma 3.3. — *A subanalytic (and continuous) function in one variable $f \circ \gamma$ is not analytic at $0 \in \mathbb{R}$ for one of the following two reasons:*

i) *Puiseux expansion $f \circ \gamma(t) = \sum_{\nu=0}^{\infty} a_\nu t^{\nu/r}$, $t > 0$ contains a nonzero term with a fractional exponent. Hence $f \circ \gamma(t)$, $t > 0$ cannot be extended analytically through $0 \in \mathbb{R}$. Clearly, the same obstruction may come from extending of $f \circ \gamma(t)$, $t < 0$.*

ii) *Both functions $g_+ = f \circ \gamma(t)$, $t > 0$ and $g_- = f \circ \gamma(t)$, $t < 0$ have analytic extensions through 0, but the extensions of g_+ and g_- are not equal.*

Proof. — Immediate from the existence of Puiseux expansions for g_+ and g_-. □

The main tool in the proof of our theorem is the *Rectilinearization of subanalytic functions* due to Parusiński [**16**], [**15**]. In fact, this is a stronger version of Hironaka's *Rectilinearization Theorem* ([**7**], see also [**1**]). For the reader's convenience we recall it here.

***Theorem 3.4* (Parusiński [16])**. — *Let $f : U \to \mathbb{R}$ be a bounded continuous subanalytic function defined in an open subset of $\mathbb{R}^n$. Then there exists a locally finite collection Ψ of real analytic morphisms $\phi_\alpha : W_\alpha \to \mathbb{R}^n$ such that:*

i) *each* W_α *is isomorphic to* $\mathbb{R}^n$ *and there are compact subsets* $K_\alpha \subset W_\alpha$ *such that*

$$\bigcup \phi_\alpha(K_\alpha) = \overline{U}$$

· ii) *for each* α*, there exists* $r_i \in \mathbb{N}, i = 1, \dots, n$*, such that*

$$\phi_\alpha = \sigma_\alpha \circ \psi_\alpha,$$

where $\sigma_\alpha : V_\alpha \to \mathbb{R}^n, V_\alpha$ *isomorphic to* $\mathbb{R}^n$*, is the composition of finite sequence of local blowings up with smooth centres and*

$$\psi_\alpha = (\varepsilon_1 x_1^{r_1}, \varepsilon_2 x_2^{r_2}, \dots, \varepsilon_n x_n^{r_n}), \text{for some } \varepsilon_i = -1 \text{ or } 1. \tag{3.1}$$

iii) *for each* α*,* $\phi_\alpha(W_\alpha) \subset \overline{U}$*, and* $f \circ \phi_\alpha$ *extends from* $\phi_\alpha^{-1}(U)$ *on* W_α *to one of the following functions:*

a) the function identically equal to zero,

b) a normal crossings.

iv) *if* $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha \in \Psi$ *and* $\phi_\alpha(0) \in U$*, then* $\phi_\alpha(W_\alpha) \subset U$ *and for each* ψ *as in* (3.1) *(i.e. with all possible* ε_i*, but fixed* r_i*), the composition* $\sigma_\alpha \circ \psi \in \Psi$*.*

Remark 3.5. — The original statement of Theorem 2.7 in [**16**] contains an inaccuracy: at (i) it is claimed that $\bigcup \phi_\alpha(K_\alpha)$ is a neighbourhood of $\overline{U}$, but in fact the family $\phi_\alpha(K_\alpha)$ is only a covering of $\overline{U}$. However the set $\bigcup \sigma_\alpha(K_\alpha)$ is actually a neighbourhood of $\overline{U}$. Note that, as stated in theorem 2.7 in [**16**], in the claim (iii) we have also the third possibility, namely that $f \circ \phi_\alpha$ extends to an inverse of normal crossing. But this will not happen in our case since we consider only bounded functions.

We consider now a composed function $g_\alpha = f \circ \sigma_\alpha : \sigma_\alpha^{-1}(U) \to \mathbb{R}$. Let Q_α be an open quadrant in $V_\alpha = \mathbb{R}^n$. Note that by (iii) in the above theorem the function $g_\alpha = f \circ \sigma_\alpha$ extends analytically on Q_α. For simplicity we denote this extension again by g_α, observe that this extension is subanalytic.

We will study the arc-analyticity of our subanalytic function $g_\alpha : Q_\alpha \to \mathbb{R}$ also at the points of the boundary of Q_α. To this end we denote by $S^+_{\text{naa}}(g_\alpha)$ the set of points $x \in V_\alpha$, such that there exists an analytic arc

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow V_\alpha, \quad \gamma(0) = x, \ \gamma(0, \varepsilon) \subset Q_\alpha,$$

and such that $g_\alpha \circ \gamma(t)$, $t > 0$, cannot be extended analytically on $(-\varepsilon', \varepsilon)$, for any $\varepsilon' > 0$.

We have the following lemma.

Lemma 3.6. — *The set* $S^+_{\text{naa}}(g_\alpha)$ *is a closed subanalytic, nowhere dense, subset of* V_α*.*

Proof. — Clearly $S^+_{\text{naa}}(g_\alpha) \subset \overline{Q_\alpha} \setminus Q_\alpha$. We may assume that Q_α is the set $\{x_i > 0 \mid i = 1, \dots, n\}$. Recall that g_α is analytic on this quadrant, hence $S^+_{\text{naa}}(g_\alpha)$ will be contained in its boundary.

By Theorem 3.4, there are integers $r_i \in \mathbb{N}$, $i = 1, \dots, n$, such that

$$h_\alpha = g_\alpha(x_1^{r_1}, x_2^{r_2}, \dots, x_n^{r_n}), \tag{3.2}$$

extends to an analytic function on $W_\alpha = \mathbb{R}^n$. Let us denote by H_i the hyperplane $\{x_i = 0\}$. Since our function g_α is analytic on the first quadrant $\{x_i > 0 \mid i = 1, \dots, n\}$, then clearly we have $S^+_{\mathrm{naa}}(g_\alpha) \subset \bigcup_{i=1}^n \overline{H}_i^+$, where $H_i^+ = \{x \in H_i;\, x_j > 0,\, j \in \{1, \dots, n\} \smallsetminus \{i\}\}$ is an open quadrant in H_i. Let us consider fixed quadrant $\overline{H}_1^+ = \{x_1 = 0;\, x_j \geqslant 0,\, j \in \{2, \dots, n\}\}$. Now we have a Puiseux expansion (which follows from (3.2))

$$g_\alpha(x_1, x') = \sum_{\nu=0}^{\infty} a_\nu(x') x_1^{\nu/r_1}, \quad \nu, r_1 \in \mathbb{N}, \tag{3.3}$$

for $x' = (x_2, \dots, x_n)$ and x_1 such that $x_i > 0$, $i = 1, \dots, n$. Moreover, a_ν are analytic functions in H_1^+ such that $a_\nu(x_2^{r_2}, \dots, x_n^{r_n})$ extend to analytic functions.

Let $(0, x') \in H_1^+$. The following observations are immediate consequences of (3.3):

i) if there is an open (in H_1^+) neighbourhood Ω of x' such that $a_\nu = 0$ in Ω, for all $\nu \in \mathbb{N} \smallsetminus r_1\mathbb{N}$, then $x' \notin S^+_{\mathrm{naa}}(g_\alpha)$, (more precisely $\Omega \cap S^+_{\mathrm{naa}}(g_\alpha) = \varnothing$).

ii) if there exists $\nu_0 \in \mathbb{N} \smallsetminus r_1\mathbb{N}$ such that $a_{\nu_0}(x') \neq 0$, then g_α cannot be extended, through $x_1 = 0$, on the arc (linear segment) $x_1 \to (x_1, x')$. Therefore then $(0, x') \in S^+_{\mathrm{naa}}(g_\alpha)$.

Observe that in the first case we may assume that $\Omega = H_1^+$, since all a_ν are analytic functions in H_1^+. So in this case $H_1^+ \cap S^+_{\mathrm{naa}}(g_\alpha) = \varnothing$.

So we are left with the second case. We shall prove that

$$H_1 \cap S^+_{\mathrm{naa}}(g_\alpha) = H_1 \cap \overline{Q_\alpha}. \tag{$*$}$$

Note that here we are in the hyperplane H_1 and not in the open quadrant H_1^+. By i), ii) and $(*)$ it follows that $S^+_{\mathrm{naa}}(g_\alpha)$ is closed and subanalytic.

To prove $(*)$ we denote by ν_0 the smallest $\nu \in \mathbb{N} \smallsetminus r_1\mathbb{N}$ such that $a_\nu \not\equiv 0$ in H_1^+. Let $(0, x') \in \overline{H}_1^+$, if $a_{\nu_0}(x') \neq 0$, then by ii), $(0, x') \in S^+_{\mathrm{naa}}(g_\alpha)$. Assume that $a_{\nu_0}(x') = 0$. Let $\eta(t)$, $t \in (-\varepsilon, \varepsilon)$ be an analytic arc in H_1 such that $\eta(0) = x'$ and $\eta(t) \in H_1^+$, $a_{\nu_0}(\eta(t)) \neq 0$ for $t \in (0, \varepsilon)$. Let r be the smallest common multiple of $r_2, \dots, r_n$. By (ii) of Theorem 3.4 it follows that $a_\nu(\eta(t^r))$ is analytic at $0 \in \mathbb{R}$, for any $\nu \in \mathbb{N}$. For simplicity we denote again $\eta(t^r)$ by $\eta(t)$.

We are going to choose a suitable exponent $N \in \mathbb{N}$ such that on the arc

$$\gamma(t) = (t^N, \eta(t)), \quad t > 0,$$

the function g_α cannot be analytically extended through 0. Note that, if we substract in (3.3), all terms $a_\nu(x') x_1^{\nu/r_1}$ with $\nu < \nu_0$, the set $S^+_{\mathrm{naa}}(g_\alpha)$ remains the same (indeed all these terms are analytic in V_α). So we may assume that in (3.3) we have only terms for $\nu \geqslant \nu_0$. Hence we obtain the Puiseux expansion

$$g_\alpha(t^N, \eta(t)) = \sum_{\nu=\nu_0}^{\infty} a_\nu(\eta(t)) t^{\nu N/r_1}, \quad t > 0. \tag{3.4}$$