Séminaires & Congrès
10, 2005, p. 191–208

UNIMODAL SINGULARITIES AND DIFFERENTIAL OPERATORS

by

Yayoi Nakamura & Shinichi Tajima

Abstract. — An algebraic local cohomology class attached to a hypersurface isolated singularity is considered from the view point of algebraic analysis. A holonomic system derived from first order differential equations associated to a cohomology class and its solutions are studied. For the unimodal singularities case, it is shown that the multiplicity of the holonomic system associated to the cohomology class, which generates the dual space of Milnor algebra, is equal to two.

***Résumé* (Singularités unimodulaires et opérateurs différentiels).** — On considère une classe de cohomologie locale algébrique attachée à une hypersurface à singularités isolées, du point de vue de l'analyse algébrique. On étudie le système holonome des équations aux dérivées partielles du premier ordre associé à la classe de cohomologie ainsi que ses solutions. On décrit une méthode générale pour examiner le système holonome associé. Il est montré que, dans le cas de singularités isolées unimodales, la multiplicité du système holonome associé à la classe génératrice de l'espace dual de l'algèbre de Milnor est égale à deux. Une description explicite des solutions du système holonome est donnée.

1. Introduction

We consider algebraic local cohomology classes attached to hypersurface isolated singularities by using first order differential operators. The purpose is to clarify the difference between quasihomogeneity and non-quasihomogeneity of the sinuglarity from a view point of $\mathcal{D}$-modules theory.

In [**3**], we gave a characterization of quasihomogeneity of hypersurface isolated singularities based on $\mathcal{D}$-modules theory. We considered an algebraic local cohomology class attached to a given singularity which generates the dual space of Milnor algebra, and an associated holonomic system derived from first order annihilators of the cohomology class in consideration. We showed that the simplicity of the associated

2000 ***Mathematics Subject Classification.*** — Primary 32S25; Secondary 32C36, 32C38.
Key words and phrases. — Isolated hypersurface singularities, algebraic local cohomology classes, holonomic $\mathcal{D}$-modules.

first order holonomic system is equivalent to the quasihomogeneity of the singularity. For non-quasihomogeneous case, the structure of associated first order holonomic system is not fully investigated. By putting the idea in [**3**] to practical use to non-quasihomogeneous singularities, we consider relations between non-quasihomogeneous singularities and a structure of first order holonomic systems. In this paper, we give a practical method of computations. Applying the method, we give an explicit description of algebraic local cohomology solution space of the holonomic system in question and also present a detailed result of computations on normal forms of exceptional unimodal singularities.

In § 2, we briefly recall the dual space of Milnor algebra with respect to the Grothendieck local duality and introduce a holonomic system derived from differential operators of order at most one which annihilate a generator of the dual space. In § 3, we give a method for describing the solution space of the first order holonomic system. We recall our results on the quasihomogeneous singularities and the unimodal singularities concerning to the solution space of the holonomic system. We show, for the unimodal singularities case, that the solution space of the holonomic system derived from first order differential equations is of dimension two. In § 4, we give a method for examining semiquasihomogeneous singularities from the computational point of view. We show that the computation of the solution space of the first order holonomic system can be carried out in finite dimensional vector spaces. In § 5, we give results of the computations for each normal form of exceptional unimodal singularities. For proves of results stated in § 2 and § 3, please refer to [**3**].

2. The dual space of Milnor algebra and first order differential operators

Let X be an open neighborhood of the origin O in the n dimensional affine space $\mathbb{C}^n$ and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Let $f(z) \in \mathcal{O}_{X,O}$ be a holomorphic function on X with an isolated singularity at the origin O. Denote by $\mathcal{I}$ the ideal in $\mathcal{O}_{X,O}$ generated by the partial derivatives $f_j = \partial f(z)/\partial z_j$ $(j = 1, \dots, n)$ of $f(z)$:

$$\mathcal{I} = \langle f_1, \dots, f_n \rangle_O.$$

From the Grothendieck local duality, we have a non-degenerate perfect pairing

$$\Omega^n_{X,O}/\mathcal{I}\Omega^n_{X,O} \times \mathcal{E}xt^n_{\mathcal{O}_{X,O}}(\mathcal{O}_{X,O}/\mathcal{I}, \mathcal{O}_{X,O}) \to \mathbb{C} \tag{1}$$

where Ω^n_X is the sheaf of holomorphic differential n-forms on X. Let Σ be the space of algebraic local cohomology classes annihilated by the ideal $\mathcal{I}$:

$$\Sigma = \{\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid g(z)\eta = 0,\, g(z) \in \mathcal{I}\}.$$

We can identify the space Σ with $\mathcal{E}xt^n_{\mathcal{O}_{X,O}}(\mathcal{O}_{X,O}/\mathcal{I}, \mathcal{O}_{X,O})$ as a finite dimensional vector spaces over $\mathbb{C}$. Then, by identifying Milnor algebra $\mathcal{O}_{X,O}/\mathcal{I}$ and $\Omega^n_{X,O}/\mathcal{I}\Omega^n_{X,O}$, we find that the space Σ is the dual space of Milnor algebra.

The space Σ is generated by a single cohomology class over $\mathcal{O}_{X,O}$. For instance, one can take the cohomology class $\sigma_f = \left[\dfrac{1}{f_1 \dots f_n}\right] \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$ as a generator over $\mathcal{O}_{X,O}$ of Σ, where the notation $\left[\dfrac{a}{b_1 \cdots b_n}\right]$ for functions a, $b_1 \cdots b_n \in \mathcal{O}_{X,O}$ stands for the algebraic local cohomology class associated to the residue symbol $\begin{bmatrix} a \\ b_1 \cdots b_n \end{bmatrix} \in \mathcal{E}xt^n_{\mathcal{O}_{X,O}}(\mathcal{O}_{X,O}/\mathcal{I}, \mathcal{O}_{X,O})$.

Let σ be a generator of Σ over $\mathcal{O}_{X,O}$:

$$\Sigma = \mathcal{O}_{X,O}\sigma.$$

Since the algebraic local cohomology group $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ has a structure of $\mathcal{D}_{X,O}$-module, we can consider annihilators of σ in $\mathcal{D}_{X,O}$ where $\mathcal{D}_X$ is the sheaf of linear partial differential operators.

Let $\mathcal{L}_f$ be the set of linear partial differential operators of order at most one which annihilate σ:

$$\mathcal{L}_f = \{P = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0,\, a_j(z) \in \mathcal{O}_{X,O},\, j = 0, 1, \dots, n\}.$$

Let $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ be the left ideal in $\mathcal{D}_{X,O}$ generated by $\mathcal{L}_f$; $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma) = \mathcal{D}_{X,O}\mathcal{L}_f$. Then $\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ defines a holonomic $\mathcal{D}_X$ module supported at the origin.

Let $P \in \mathcal{L}_f$ be a first order partial differential operator annihilating the algebraic local cohomology class σ. Such an operator has the following property :

Lemma 2.1. — *Let σ be a generator of Σ over $\mathcal{O}_{X,O}$. Let P be a first order linear partial differential operator annihilating the cohomology class σ. Then, the space Σ is closed under the action of P, i.e., $P(\Sigma) \subseteq \Sigma$.*

It is obvious that the condition whether a given first order differential operator acts on Σ or not depends only on its first order part. We introduce Θ_f to be the set of differential operators of the form $\sum_{j=1}^n a_j(z)\dfrac{\partial}{\partial z_j}$ acting on Σ. Then, an operator v is in Θ_f if and only if v satisfies the condition $vg(z) \in \mathcal{I}$ for every $g(z) \in \mathcal{I}$, i.e.,

$$\Theta_f = \{v = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} \mid vg(z) \in \mathcal{I},^{\forall}\, g(z) \in \mathcal{I},\, a_j(z) \in \mathcal{O}_{X,O},\, j = 1, \dots, n\}.$$

Lemma 2.2. — *The mapping, from $\mathcal{L}_f$ to Θ_f, which associates the first order part $v_p \in \Theta_f$ to $P \in \mathcal{L}_f$ is a surjective mapping.*

3. Solution space of the holonomic system

Let $\sigma \in \Sigma$ be a generator of Σ over $\mathcal{O}_{X,O}$. Let η be another algebraic local cohomology class in Σ and $h(z) \in \mathcal{O}_{X,O}$ a holomorphic function satisfying $\eta = h(z)\sigma$.

It is obvious that, to represent $\eta \in \Sigma$ in the form $\eta = h\sigma$, it suffices to take the modulo class $h \mod \mathcal{I}$ of the holomorphic function $h(z) \in \mathcal{O}_{X,O}$. Let P be an annihilator of σ in $\mathcal{L}_f$. Now let us consider the condition that an algebraic local cohomology class $\eta \in \Sigma$ becomes a solution of homogeneous differential equation $P\eta = 0$. An element $v \in \Theta_f$ induces a linear operator acting on $\mathcal{O}_{X,O}/\mathcal{I}$ which is also denoted by v:

$$v : \mathcal{O}_{X,O}/\mathcal{I} \to \mathcal{O}_{X,O}/\mathcal{I}.$$

For the first order part $v_P = \sum_{j=1}^n a_j(z)\partial/\partial z_j \in \Theta_f$ of an annihilator P, we have

$$v_P h = \sum_{j=1}^{n} a_j(z) \frac{\partial h}{\partial z_j} = 0 \mod \mathcal{I}.$$

Let $\mathcal{H}_f$ be a set of modulo classes by $\mathcal{I}$ of functions $h(z)$ that satisfies $vh(z) \in \mathcal{I}$ for $v \in \Theta_f$:

$$\mathcal{H}_f = \{h \in \mathcal{O}_{X,O}/\mathcal{I} \mid vh = 0, \ ^\forall v \in \Theta_f\}.$$

Concerning to the algebraic local cohomology solutions of the holonomic system $\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$, we have the following result ([**3**]):

Theorem 3.1. — *Let $f(z)$ be a function defining an isolated singularity at the origin. Let $\sigma \in \Sigma$ be a generator over $\mathcal{O}_{X,O}$ of Σ. Then,*

$$Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \{h\sigma \mid h \in \mathcal{H}_f\}.$$

The space $\mathcal{H}_f$ does not depend on a choice of a generator σ of Σ. Actually, the space $\mathcal{H}_f$ is completely determined by the ideal $\mathcal{I}$. Thus, in this sence, the space $\mathcal{H}_f$ is an intrinsic object in the study of the solution space $Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X))$.

For the quasihomogeneous isolated singularity case, we have the following result ([**3**]) :

Proposition 3.2. — *Let $f(z)$ be a function defining a quasihomogeneous isolated singularity at the origin. Then*

$$\mathcal{H}_f = \mathrm{Span}_{\mathbb{C}}\{1\}.$$

Let $\mathcal{A}nn_{\mathcal{D}_{X,O}}(\sigma)$ be the left ideal in $\mathcal{D}_{X,O}$ of annihilators of the cohomology class σ.

***Theorem 3.3* ([3])**. — *Let $f = f(z)$ be a function defining an isolated singularity at the origin O and σ a generator of Σ. The following three conditions are equivalent :*

(i) $\mathcal{O}_{X,O}\langle f, f_1, \cdots, f_n\rangle = \mathcal{O}_{X,O}\langle f_1, \cdots, f_n\rangle$.

(ii) $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma) = \mathcal{A}nn_{\mathcal{D}_{X,O}}(\sigma)$.

(iii) $Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}_{\mathbb{C}}\{\sigma\}$.

This result can be regarded as a counterpart in the algebraic local cohomology theory of a result by K.Saito on a characterization of quasihomogeneity of singularities ([**5**]).

In contrast, for a non-quasihomogeneous function $f = f(z)$,

$$\mathcal{O}_{X,O}\langle f_1, \ldots, f_n\rangle \neq \mathcal{O}_{X,O}\langle f, f_1, \ldots, f_n\rangle \tag{2}$$

and thus

$$\dim \mathit{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\,\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \geqslant 2.$$

It seems natural to expect that the solution space

$$\mathit{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\,\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X))$$

is related to non-quasihomogeneity of a given hypersurface isolated singularity. Let us consider the structure of the solution spaces for exceptional unimodal singularities which are most typical non-quasihomogeneous singularities.

We have the following result :

Proposition 3.4. — *For a function $f(z)$ defining an exceptional unimodal singularity,*

$$\mathcal{H}_f = \mathrm{Span}_{\mathbb{C}}\{1, f \bmod \mathcal{I}\}.$$

Proposition 3.4 is proved by direct computations for each normal form of exceptional unimodal singularities. We shall explain a method we used for computations in the next section.

We arrive at the following theorem ([**6**]):

Theorem 3.5. — *Let $f(z)$ be a function defining an exceptional unimodal singularity. Then,*

$$\mathit{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\,\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}_{\mathbb{C}}\{\sigma, \delta\},$$

where δ is the delta function with support at the origin O.

Proof. — Theorem 3.1 together with Proposition 3.4 yields that the solution space $\mathit{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\,\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X))$ is spanned by σ and $f\sigma$. Since the ideal quotient $\mathcal{I} : \langle f\rangle$ is the maximal ideal m in $\mathcal{O}_{X,O}$ for any exceptional unimodal singularities, the cohomology class $f\sigma$ is annihilated by m. This implies that $f\sigma = \mathrm{const}.\delta$ where $\delta = \left[\dfrac{1}{z_1 \cdots z_n}\right]$. It completes the proof. □

We note here that it is possible to characterize the cohomology class σ attached to an exceptional unimodal (and bimodal) singularity as the solution of a *second order* holonomic system. We shall treat this subject elsewhere.