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A SURVEY ON ALEXANDER POLYNOMIALS OF PLANE CURVES

by

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Abstract. — In this paper, we give a brief survey on the fundamental group of the complement of a plane curve and its Alexander polynomial. We also introduce the notion of θ-Alexander polynomials and discuss their basic properties.

Résumé **(Un état des lieux sur les polynômes d'Alexander des courbes planes)**
Dans cet article, nous donnons un bref état des lieux sur le groupe fondamental du complémentaire d'une courbe plane et son polynôme d'Alexander. Nous introduisons de plus la notion de polynôme d'Alexander de type θ et discutons leurs propriétés élémentaires.

1. Introduction

For a given hypersurface $V \subset \mathbf{P}^n$, the fundamental group $\pi_1(\mathbf{P}^n - V)$ plays a crucial role when we study geometrical objects over $\mathbf{P}^n$ which are branched over V. By the hyperplane section theorem of Zariski [**51**], Hamm-Lê [**16**], the fundamental group $\pi_1(\mathbf{P}^n - V)$ can be isomorphically reduced to the fundamental group $\pi_1(\mathbf{P}^2 - C)$ where $\mathbf{P}^2$ is a generic projective subspace of dimension 2 and $C = V \cap \mathbf{P}^2$. A systematic study of the fundamental group was started by Zariski [**50**] and further developments have been made by many authors. See for example Zariski [**50**], Oka [**31–33**], Libgober [**22**]. For a given plane curve, the fundamental group $\pi_1(\mathbf{P}^2 - C)$ is a strong invariant but it is not easy to compute. Another invariant which is weaker but easier to compute is the Alexander polynomial $\Delta_C(t)$. This is related to a certain infinite cyclic covering space branched over C. Important contributions are done by Libgober, Randell, Artal, Loeser-Vaquié, and so on. See for example [**1, 2, 7, 9, 10, 13, 14, 20, 24, 26, 29, 41, 43, 44, 46, 47**]

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The main purpose of this paper is to give a survey for the study of the fundamental group and the Alexander polynomial (§§ 2,3). However we also give a new result on θ-Alexander polynomials in section 4.

In section two, we give a survey on the fundamental group of the complement of plane curves. In section three, we give a survey for the Alexander polynomial. It turns out that the Alexander polynomial does not tell much about certain non-irreducible curves. A possibility of a replacement is *the characteristic variety* of the multiple cyclic covering. This theory is introduced by Libgober [**23**].

Another possibility is *the Alexander polynomial set* (§ 4). For this, we consider the infinite cyclic coverings branched over C which correspond to the kernel of arbitrary surjective homomorphism $\theta : \pi_1(\mathbf{C}^2 - C) \to \mathbf{Z}$ and we consider the *θ-Alexander polynomial.* Basic properties are explained in the section 4.

2. Fundamental groups

The description of this section is essentially due to the author's lecture at School of Singularity Theory at ICTP, 1991.

2.1. van Kampen Theorem.— Let $C \subset \mathbf{P}^2$ be a projective curve which is defined by $C = \{[X,Y,Z] \in \mathbf{P}^2 \mid F(X,Y,Z) = 0\}$ where $F(X,Y,Z)$ is a reduced homogeneous polynomial $F(X,Y,Z)$ of degree d. The first systematic studies of the fundamental group $\pi_1(\mathbf{P}^2 - C)$ were done by Zariski [**49–51**] and van Kampen [**18**]. They used so called *pencil section method* to compute the fundamental group. This is still one of the most effective method to compute the fundamental group $\pi_1(\mathbf{P}^2 - C)$ when C has many singularities.

Let $\ell(X,Y,Z)$, $\ell'(X,Y,Z)$ be two independent linear forms. For any $\tau = (S,T) \in \mathbf{P}^1$, let $L_\tau = \{[X,Y,Z] \in \mathbf{P}^2 \mid T\ell(X,Y,Z) - S\ell'(X,Y,Z) = 0\}$. The family of lines $\mathcal{L} = \{L_\tau \mid \tau \in \mathbf{P}^1\}$ is called the *pencil* generated by $L = \{\ell = 0\}$ and $L' = \{\ell' = 0\}$. Let $\{B_0\} = L \cap L'$. Then $B_0 \in L_\tau$ for any τ and it is called *the base point* of the pencil. We assume that $B_0 \notin C$. L_τ is called a *generic line* (resp. *non-generic line*) of the pencil for C if L_τ and C meet transversally (resp. non-transversally). If L_τ is not generic, either L_τ passes through a singular point of C or L_τ is tangent to C at some smooth point. We fix two generic lines L_{τ_0} and L_{τ_∞}. Hereafter we assume that τ_∞ is the point at infinity ∞ of $\mathbf{P}^1$ (so $\tau_\infty = \infty$) and we identify $\mathbf{P}^2 - L_\infty$ with the affine space $\mathbf{C}^2$. We denote the affine line $L_\tau - \{B_0\}$ by L_τ^a. Note that $L_\tau^a \cong \mathbf{C}$. The complement $L_{\tau_0} - L_{\tau_0} \cap C$ (resp. $L_{\tau_0}^a - L_{\tau_0}^a \cap C$) is topologically S^2 minus d points (resp. $(d+1)$ points). We usually take $b_0 = B_0$ as the base point in the case of $\pi_1(\mathbf{P}^2 - C)$. In the affine case $\pi_1(\mathbf{C}^2 - C)$, we take the base point b_0 on L_{τ_0} which is sufficiently near to B_0 but $b_0 \neq B_0$. Let us consider two free groups

$$F_1 = \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \quad \text{and} \quad F_2 = \pi_1(L_{\tau_0}^a - L_{\tau_0}^a \cap C, b_0).$$

of rank $d-1$ and d respectively. We consider the set

$$\Sigma := \{\tau \in \mathbf{P}^1 \mid L_\tau \text{ is a non-generic line}\} \cup \{\infty\}.$$

We put ∞ in Σ so that we can treat the affine fundamental group simultaneously. We recall the definition of the action of the fundamental group $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ on F_1 and F_2. We consider the blowing up $\widetilde{\mathbf{P}^2}$ of $\mathbf{P}^2$ at B_0. $\widetilde{\mathbf{P}^2}$ is canonically identified with the subvariety

$$W = \{((X,Y,Z),(S,T)) \in \mathbf{P}^2 \times \mathbf{P}^1 \mid T\ell(X,Y,Z) - S\ell'(X,Y,Z) = 0\}$$

through the first projection $p : W \to \mathbf{P}^2$. Let $q : W \to \mathbf{P}^1$ be the second projection. The fiber $q^{-1}(s)$ is canonically isomorphic to the line L_s. Let $E = \{B_0\} \times \mathbf{P}^1 \subset W$. Note that E is the exceptional divisor of the blowing-up $p : W \to \mathbf{P}^2$ and $q|_E : E \to \mathbf{P}^1$ is an isomorphism. We take a tubular neighbourhood N_E of E which can be identified with the normal bundle of E. As the projection $q|N_E \to \mathbf{P}^1$ gives a trivial fibration over $\mathbf{P}^1 - \{\infty\}$, we fix an embedding $\phi : \Delta \times (\mathbf{P}^1 - \{\infty\}) \to N_E$ such that $\phi(0,\eta) = (B_0,\eta)$, $\phi(1,\tau_0) = (b_0,\tau_0)$ and $q(\phi(t,\eta)) = \eta$ for any $\eta \in \mathbf{P}^1 - \{\infty\}$. Here $\Delta = \{t \in \mathbf{C}; |t| \leqslant 1\}$. In particular, this gives a section of q over $\mathbf{C} = \mathbf{P}^1 - \{\infty\}$ by $\eta \mapsto b_{0,\eta} := \phi(1,\eta) \in L^a_\eta$. We take $b_{0,\eta}$ as the base point of the fiber L^a_η. Let $\widetilde{C} = p^{-1}(C)$. The restrictions of q to $\widetilde{C}$ and $\widetilde{C} \cup E$ are locally trivial fibrations by Ehresman's fibration theorem [**48**]. Thus the restrictions $q_1 := q|_{(W-\widetilde{C})}$ and $q_2 := q|_{(W-\widetilde{C}\cup E)}$ are also locally trivial fibrations over $\mathbf{P}^1 - \Sigma$. The generic fibers of q_1, q_2 are homeomorphic to $L_{\tau_0} - C$ and $L^a_{\tau_0} - C$ respectively. Thus there exists canonical action of $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ on F_1 and F_2. We call this action *the monodromy action* of $\pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$. For $\sigma \in \pi_1(\mathbf{P}^1 - \Sigma, \tau_0)$ and $g \in F_1$ or F_2, we denote the action of σ on g by g^σ. The relations in the group F_ν

$$(R_1) \qquad \langle g^{-1}g^\sigma = e \mid g \in F_\nu, \sigma \in \pi_1(\mathbf{P}^1 - \Sigma, \tau_0)\rangle, \quad \nu = 1,2$$

are called *the monodromy relations*. The normal subgroup of F_ν, $\nu = 1,2$ which are normally generated by the elements $\{g^{-1}g^\sigma, \mid g \in F_\nu\}$ are called *the groups of the monodromy relations* and we denote them by N_ν for $\nu = 1,2$ respectively. The original van Kampen Theorem can be stated as follows. See also [**5, 6**].

Theorem 1 **([18])**. — *The following canonical sequences are exact.*

$$1 \to N_1 \to \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \to \pi_1(\mathbf{P}^2 - C, b_0) \to 1$$
$$1 \to N_2 \to \pi_1(L^a_{\tau_0} - L^a_{\tau_0} \cap C, b_0) \to \pi_1(\mathbf{C}^2 - C, b_0) \to 1$$

Here 1 is the trivial group. Thus the fundamental groups $\pi_1(\mathbf{P}^2 - C, b_0)$ and $\pi_1(\mathbf{C}^2 - C, b_0)$ are isomorphic to the quotient groups F_1/N_1 and F_2/N_2 respectively.

For a group G, we denote the commutator subgroup of G by $D(G)$. The relation of the fundamental groups $\pi_1(\mathbf{P}^2 - C, b_0)$ and $\pi_1(\mathbf{C}^2 - C, b_0)$ are described by the following. Let $\iota : \mathbf{C}^2 - C \to \mathbf{P}^2 - C$ be the inclusion map.

Lemma 2 **([30])**. — *Assume that L_∞ is generic.*

(1) *We have the following central extension.*

$$1 \longrightarrow \mathbf{Z} \xrightarrow{\gamma} \pi_1(\mathbf{C}^2 - C, b_0) \xrightarrow{\iota_\#} \pi_1(\mathbf{P}^2 - C, b_0) \longrightarrow 1$$

A generator of the kernel Ker $\iota_\#$ *of* $\iota_\#$ *is given by a lasso* ω *for* L_∞.

(2) *Furthermore, their commutator subgroups coincide i.e.,* $D(\pi_1(\mathbf{C}^2 - C)) = D(\pi_1(\mathbf{P}^2 - C))$.

Proof. — A loop ω is called *a lasso* for an irreducible curve D if ω is homotopic to a path written as $\ell \circ \tau \circ \ell^{-1}$ where τ is the boundary circle of a normal small disk of D at a smooth point and ℓ is a path connecting the base point and τ [**35**]. For the assertion (1), see [**30**]. We only prove the second assertion. Assume that C has r irreducible components of degree $d_1, \dots, d_r$. The restriction of the homomorphism $\iota_\#$ gives a surjective morphism $\iota_\# : D(\pi_1(\mathbf{C}^2 - C)) \to D(\pi_1(\mathbf{P}^2 - C))$. If there is a $\sigma \in$ Ker $\iota_\# \cap D(\pi_1(\mathbf{C}^2 - C))$, σ can be written as $\gamma(\omega)^a$ for some $a \in \mathbf{Z}$. As ω corresponds to $(d_1, \dots, d_r)$ in the homology $H_1(\mathbf{C}^2 - C) \cong \mathbf{Z}^r$, σ corresponds to $(ad_1, \dots, ad_r)$. As σ is assumed to be in the commutator group, this must be trivial. That is, $a = 0$. □

2.2. Examples of monodromy relations. — We recall several basic examples of the monodromy relations. Let C be a reduced plane curve of degree d.

We consider a model curve $C_{p,q}$ which is defined by $y^p - x^q = 0$ and we study $\pi_1(\mathbf{C}^2 - C_{p,q})$. For this purpose, we consider the pencil lines $x = t$, $t \in \mathbf{C}$. We consider the local monodromy relations for σ, which is represented by the loop $x = \varepsilon(2\pi i t)$, $0 \leqslant t \leqslant 1$. We take local generators $\xi_0, \xi_1, \dots, \xi_{p-1}$ of $\pi_1(L_\varepsilon, b_0))$ as in Figure 1. Every loops are counter-clockwise oriented. It is easy to see that each point of $C_{p,q} \cap L_\varepsilon$ are rotated by the angle $2\pi \times q/p$. Let $q = mp + q'$, $0 \leqslant q' < p$. Then the monodromy relations are:

$$(R_1) \qquad \xi_j (= \xi_j^\sigma) = \begin{cases} \omega^m \xi_{j+q'} \omega^{-m}, & 0 \leqslant j < p - q' \\ \omega^{m+1} \xi_{j+q'-p} \omega^{-(m+1)}, & p - q' \leqslant j \leqslant p-1 \end{cases}$$

$$(R_2) \qquad \omega = \xi_{p-1} \cdots \xi_0.$$

The last relation in (R_1) can be omitted as it follows from the other relations.

$$\begin{aligned} \xi_{p-1} &= \omega(\xi_{p-2} \cdots \xi_0)^{-1} \\ &= \omega \omega^m \xi_{q'}^{-1} \omega^{-m} \dots \omega^m \xi_{p-1}^{-1} \omega^{-m} \omega^{m+1} \xi_0^{-1} \omega^{-m-1} \cdots \omega^{m+1} \xi_{q'-2}^{-1} \omega^{-m-1} \\ &= \omega^{m+1} \xi_{q'-1} \omega^{-m-1}. \end{aligned}$$

For the convenience, we introduce two groups $G(p,q)$ and $G(p,q,r)$.

$$G(p,q) := \langle \xi_1, \dots, \xi_p, \omega \mid R_1, R_2 \rangle, \quad G(p,q,r) := \langle \xi_1, \dots, \xi_p, \omega \mid R_1, R_2, R_3 \rangle$$

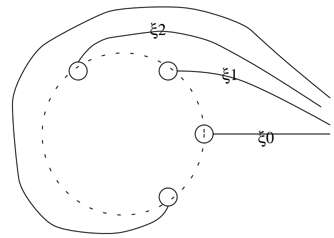

FIGURE 1. Generators

where R_3 is the vanishing relation of the big circle $\partial D_R = \{|y| = R\}$:

$$(R_3) \qquad \omega^r = e.$$

Now the above computation gives the following.

Lemma 3. — *We have* $\pi_1(\mathbf{C}^2 - C_{p,q}, b_0) \cong G(p,q)$ *and* $\pi_1(\mathbf{P}^2 - C_{p,q}, b_0) \cong G(p,q,1)$.

The groups of $G(p,q)$ and $G(p,q,r)$ are studied in [**12, 32**]. For instance, we have

***Theorem 4* ([32])**

(i) *Let* $s = \gcd(p,q)$, $p_1 = p/s$, $q_1 = q/s$. *Then* ω^{q_1} *is the center of* $G(p,q)$.

(ii) *Put* $a = \gcd(q_1, r)$. *Then* ω^a *is in the center of* $G(p,q,r)$ *and has order* r/a *and the quotient group* $G(p,q,r)/<\omega^a>$ *is isomorphic to* $\mathbf{Z}_{p/s} * \mathbf{Z}_a * F(s-1)$.

***Corollary 5* ([32])**. — *Assume that* $r = q$. *Then* $G(p,q,q) = \mathbf{Z}_{p_1} * \mathbf{Z}_{q_1} * F(s-1)$. *In particular, if* $\gcd(p,q) = 1$, $G(p,q,q) \cong \mathbf{Z}_p * \mathbf{Z}_q$.

Let us recall some useful relations which follow from the above model.

(I) Tangent relation. — Assume that C and L_0 intersect at a simple point P with intersection multiplicity p. Such a point is called a *flex* point of order $p-2$ if $p \geqslant 3$ ([**50**]). This corresponds to the case $q = 1$. Then the monodromy relation gives $\xi_0 = \xi_1 = \cdots = \xi_{p-1}$ and thus $G(p,1) \cong \mathbf{Z}$. As a corollary, Zariski proves that *the fundamental group* $\pi_1(\mathbf{P}^2 - C)$ *is abelian if* C *has a flex of order* $\geqslant d-3$. In fact, if C has a flex of order at least $d-3$, the monodromy relation is given by $\xi_0 = \cdots = \xi_{d-2}$. On the other hand, we have one more relation $\xi_{d-1} \ldots \xi_0 = e$. In particular, considering the smooth curve defined by $C_0 = \{X^d - Y^d = Z^d\}$, we get that $\pi_1(\mathbf{P}^2 - C)$ is abelian for a smooth plane curve C, as C can be joined to C_0 by a path in the space of smooth curves of degree d.

(II) Nodal relation. — Assume that C has an ordinary double point (*i.e.*, a node) at the origin and assume that C is defined by $x^2 - y^2 = 0$ near the origin. This is the case when $p = q = 2$. Then as the monodromy relation, we get the commuting relation: $\xi_1\xi_2 = \xi_2\xi_1$. Assume that C has only nodes as singularities. The commutativity of