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SYMPLECTIC 4-MANIFOLDS CONTAINING SINGULAR RATIONAL CURVES WITH $(2,3)$-CUSP

by

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Abstract. — If a symplectic 4-manifold contains a pseudo-holomorphic rational curve with a $(2,3)$-cusp of positive self-intersection number, then it must be rational.

***Résumé* (Variétés symplectiques de dimension 4 contenant des courbes rationnelles singulières avec points de rebroussement de type $(2,3)$)**
Si une variété symplectique de dimension 4 contient une courbe rationnelle pseudo-holomorphe avec un point de rebroussement de type $(2,3)$ de nombre d'auto-intersection positif, alors elle est elle-même rationnelle.

1. Introduction

In the previous paper [**5**], we studied topology of symplectic fillings of the links of simple singularities in complex dimension 2. In fact, we proved that such a symplectic filling is symplectic deformation equivalent to the corresponding Milnor fiber, if it is minimal, *i.e.*, it does not contain symplectically embedded 2-spheres of self-intersection number -1. In this short note, we present some biproduct of the argument in [**5**]. For smoothly embedded pseudo-holomorphic curves, the self-intersection number can be arbitrary large, *e.g.*, sections of ruled symplectic 4-manifolds. The situation is different for singular pseudo-holomorphic curves. In fact, we prove the following:

Main Theorem. — *Let M be a closed symplectic 4-manifold containing a pseudo-holomorphic rational curve C with a $(2,3)$-cusp point. Suppose that C is non-singular away from the $(2,3)$-cusp point. If the self-intersection number C^2 of C is positive, then M must be a rational symplectic 4-manifold and C^2 is at most 9.*

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This is a corollary of the uniqueness of symplectic deformation type of minimal symplectic fillings of the link of the simple singularity of type E_8, *i.e.*, the isolated singularity of $x^2 + y^3 + z^5 = 0$. There are similar applications of the uniqueness result for A_n and D_n cases. We can apply these results to classification of minimal symplectic fillings of quotient surface singularities other than simple singularities, which will be discussed elsewhere.

2. Preliminaries

In this section, we recall necessary materials from [**5**]. Let L be the link of an isolated surface singularity. L carries a natural contact structure ξ defined by the maximal complex tangency, *i.e.*, $\xi = TL \cap \sqrt{-1}TL$. Note that the contact structure on a $(4k+3)$-dimensional manifold induces a natural orientation on it. In particular, L, which is 3-dimensional, is naturally oriented. A compact symplectic manifold (W, ω) is called a strong symplectic filling (resp. strong concave filling) of the contact manifold (L, ξ), if the orientation of L as a contact manifold is the same as (resp. opposite to) the orientation as the boundary of a symplectic manifold W and there exists a 1-form θ on L such that $\xi = \ker\theta$ and $d\theta = \omega$. This condition is equivalent to the existence of an outward (resp. inward) normal vector field around ∂W such that $\mathcal{L}_X\omega = \omega$ and $i(X)\omega$ vanishes on ξ. Hereafter, we call strong symplectic fillings simply as symplectic fillings, since we do not use weak symplectic fillings in this note. It may be regarded as a symplectic analog of (pseudo) convexity for the boundary. Such a boundary (or a hypersurface) is said to be of contact type. Simple examples are the boundaries of convex domains, or more generally star-shaped domains in a symplectic vector space. Namely, if the convex domain contains the origin, the Euler vector field $\sum(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i})$ is a desired outward vector field. Here $\{x_i, y_i\}$ are the canonical coordinates.

Simple singularities are isolated singularities of $\mathbf{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$. Such subgroups are in one-to-one correspondence with the Dynkin diagrams of type A_n, D_n $(n \geqslant 4)$, and E_n $(n = 6, 7, 8)$.

In [**5**], we proved the following:

Theorem 2.1. — *Let X be any minimal symplectic filling of the link of a simple singularity. Then the diffeomorphism type of X is unique. Hence, it must be diffeomorphic to the Milnor fiber.*

Let us restrict ourselves to the case of type E_8 and give a sketch of the proof. Let X be a minimal symplectic filling of the link of the simple singularity of type E_8. Using Seiberg-Witten-Taubes theory, we proved that $c_1(X) = 0$, which is a special feature for the Milnor fiber and that the intersection form of X is negative definite, which is a special feature for the (minimal) resolution. In the course of the argument, we also have $b_1(X) = 0$. We glue X with another manifold Y, which is given below, to get a closed symplectic 4-manifold. To find Y, we recall K. Saito's compactification of the Milnor fiber [**6**]. The Milnor fiber $\{x^2 + y^3 + z^5 = 1\}$ is embedded in a weighted projective 3-space. Take its closure and resolve the singularities at infinity to get a smooth projective surface. Set Y a regular neighborhood of the divisor at infinity, which we call the compactifying divisor $\widetilde{D}$. Then we may assume that the boundary of Y is pseudo-concave, hence strongly symplectically concave (see Proposition 4.2). Note that the compactifying divisor consists of four rational curves with self-intersection number -1, -2, -3 and -5, respectively, which intersect one another as in Figure 2.2.

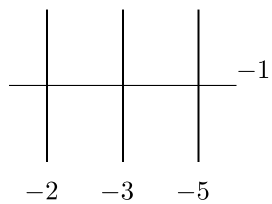

FIGURE 2.2

Topologically, the compactifying divisor $\widetilde{D}$ is the core of the plumbed manifold. We glue X and Y along boundaries to get a closed symplectic 4-manifold Z. Since $c_1(X) = 0$, $c_1(Z)$ is easily determined as the Poincaré dual of an effective divisor. In particular, we have $\int_Z c_1(Z) \wedge \omega_Z > 0$, which implies that Z is a rational or ruled symplectic 4-manifold. Note that $b_1(Z) = 0$ because of the Mayer-Vietrois sequence and the fact that $b_1(X) = 0$. Hence Z is a rational symplectic 4-manifold. Combining Hirzebruch's signature formula and calculation of the Euler number, we get $b_2(Z) = 12$. Thus Z is symplectic deformation equivalent to the 11-point blow-up of $\mathbf{C}P^2$.

The remaining task is to determine the embedding of Y in Z, or the embedding of the compactifying divisor in Z. In [**5**], we successively blow-down (-1)-curves three times to get a singular rational pseudo-holomorphic curve $\overline{D}$, see Figure 2.3.

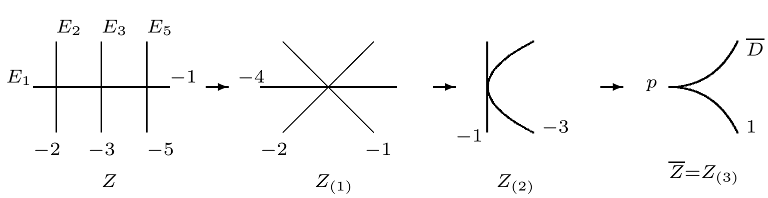

FIGURE 2.3

Then we showed that there are eight disjoint pseudo-holomorphic (-1)-curves $\{\varepsilon_i\}$ in $\overline{Z}$ so that each ε_i intersects $\overline{D}$ exactly at one point in the non-singular part of $\overline{D}$

transversally. Blowing-down ε_i, $i = 1, \ldots, 8$, we get $\mathbf{C}P^2$ and $\overline{D}$ is transformed to a singular pseudo-holomorphic cubic curve D.

Conversely, we start from a singular holomorphic cubic curve D_0 in $\mathbf{C}P^2$ with respect to the standard complex structure, *e.g.*, the one defined by $x^3 + y^2z = 0$. Pick eight points on the non-singular part of D_0 and blow up $\mathbf{C}P^2$ at these points to get $\overline{Z}_0$. Denote by $\overline{D_0}$ the proper transform of D_0. Blowing-up $\overline{Z}_0$ three more times by following the process in Figure 2.2 in the opposite way, we arrive at Z_0, the 11-point blow-up of $\mathbf{C}P^2$. It contains the total transform $\widetilde{D_0}$ of $\overline{D_0}$, which is the same configuration as in Figure 2.2. We showed, in [**5**], the following:

Theorem 2.4. — *The pair $(Z, \widetilde{D})$ is symplectic deformation equivalent to the pair $(Z_0, \widetilde{D_0})$. In particular, $\widetilde{D}$ is an anti-canonical divisor of Z.*

Recall that X is the complement of a regular neighborhood of $\widetilde{D}$ in Z, hence it is symplectic deformation equivalent to the complement of a regular neighborhood of $\widetilde{D}_0$ in Z_0. In particular, we obtained the uniqueness of symplectic deformation types.

By following the blowing-down process, we have

Corollary 2.5. — *$\overline{D}$ is an anti-canonical divisor of $\overline{Z}$.*

Note that $\overline{D}_0$ in $\overline{Z}_0$ is a holomorphic rational curve with a $(2,3)$-cusp point and that $\overline{Z}_0 \setminus \overline{D}_0 = Z_0 \setminus \widetilde{D}_0$ is a minimal symplectic filling of the simple singularity of the type E_8. We expect a similar phenomenon for our M and C in our Main Theorem. This is a key to the proof of Main Theorem.

3. Proof of Main Theorem

Let M be a closed symplectic 4-manifold and C a pseudo-holomorphic rational curve with a $(2,3)$-cusp point. Here a $(2,3)$-cusp point is defined as the singularity of $z \mapsto (z^2, z^3) + O(4)$ (see [**3**]). We assume that C is non-singular away from the cusp point. The following lemma is a direct consequence of McDuff's theorem in [**4**].

Lemma 3.1. — *C can be perturbed in a neighborhood of the cusp point so that the perturbed curve is a pseudo-holomorphic rational curve with one $(2,3)$-cusp point with respect to a tame almost complex structure, which is integrable near the cusp point.*

Proof. — Notice that $z \mapsto (z^2, z^3)$ is primitive in the sense of [**4**]. Then the conclusion follows from the proof of Theorem 2 in [**4**]. □

Remark. — The almost complex structure in the proof is not generic among tame almost complex structures, when the self-intersection number of C is less than 2.

Write $k = C^2$. Pick a tame almost complex structure on M such that C is J-holomorphic as in Lemma 3.1. If $M \setminus C$ is not minimal, we contract all J-holomorphic (-1)-rational curves which do not intersect C to get a pair (M', C) so that $M' \setminus C$

is minimal. We blow-up M' at $(k-1)$ points on the non-singular part of C to get a closed symplectic 4-manifold $\widetilde{M}$. We denote the set of the exceptional curves by $\{e_i\}$. The proper transform $\overline{D}$ of C is a pseudo-holomorphic rational curve with one $(2,3)$-cusp point and $\overline{D}^2 = 1$. Now we perform the opposite operation to the one indicated in Figure 2.3. Namely, we blow-up $\widetilde{M}$ at the cusp point of $\overline{D}$ to get two non-singular rational curves of self-intersection number -1 and -3, respectively, which are tangent to each other. These two curves are simply tangent to each other. Now we blow up the point of tangency to get three non-singular rational curves meeting at a common point pair-wisely transversally. Their self-intersection numbers are -1, -2 and -4. Finally we blow up the intersection point to get a configuration of non-singular rational curves as in Figure 2.3. This configuration is exactly the compactifying divisor $\widetilde{D}$ in section 2. We denote by N the ambient symplectic 4-manifold.

Lemma 3.2. — *The complement of a regular neighborhood of $\widetilde{D}$ in N is a symplectic filling of the link of the singularity of type E_8.*

Proof. — It is enough to see that the boundary of a regular neighborhood of $\widetilde{D}$ has a concave boundary. We can contract (-2), (-3) and (-5)-curves to get a symplectic V-manifold. The image D' of the (-1)-curve is still an embedded rational curve, whose normal bundle is of degree $-1+1/2+1/3+1/5 = 1/30 > 0$. Hence we can take a tubular neighborhood of D', whose boundary is strongly symplectically concave with the help of Darboux-Weinstein theorem [**7**]. Note that it is contactomorphic to the link of the simple singularity of type E_8. Hence the complement of a regular neighborhood of $\widetilde{D}$ in N is a symplectic filling of the link of E_8-singularity. □

Now, we show the following lemma.

Lemma 3.3. — *$N \setminus \widetilde{D}$ is minimal.*

Proof. — Assume that it is not minimal. We contract pseudo-holomorphic (-1)-rational curves f_j in $N \setminus \widetilde{D}$ to obtain $\pi : N \to \overline{N}$ such that $\overline{N} \setminus \widetilde{D}$ is minimal. Here, we use $\widetilde{D}$ for the image of $\widetilde{D}$ by π, since π is an isomorphism around $\widetilde{D}$. Then $\overline{N} \setminus \widetilde{D}$ is a minimal symplectic filling of the link of E_8-singularity. After gluing it with Y in section 2, we get back $\overline{N}$. Then Corollary 2.5 implies that $\widetilde{D} + E_1$ is an anti-canonical divisor of $\overline{N}$, which is a rational symplectic 4-manifold, where E_1 is the (-1)-curve in $\widetilde{D}$ as in Figure 2.3. Since each e_i in $\widetilde{M}$ does not contain the cusp point of $\overline{D}$, it is also a symplectic (-1)-curve in N and does not intersect E_1. By abuse of notation, we also denote it by e_i. Note that $f_j \cdot e_i \geqslant 1$ for some i, because $N \setminus \widetilde{D} \cup (\cup_i e_i)$ is minimal. On the other hand, we have

$$K_N = \pi^* K_{\overline{N}} + \sum f_j = -[\widetilde{D}] - [E_1] + \sum f_j.$$