Séminaires & Congrès
10, 2005, p. 243–254

INTEGRABILITY OF SOME FUNCTIONS ON SEMI-ANALYTIC SETS

by

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Abstract. — Using the properties of Lipschitz stratification we show that some functions on a semi-analytic sets, in particular the invariant polynomials of curvature form, are locally integrable. The result holds as well for subanalytic sets.

Résumé **(Intégrabilité de certaines fonctions sur les ensembles semi-analytiques)**
En utilisant les propriétés des stratifications lipschitziennes on montre l'intégrabilité locale d'une classe de fonctions définies sur les ensembles semi-analytiques. Cette classe contient les polynômes invariants de la courbure. Le résultat est vrai aussi pour les ensembles sous-analytiques.

I wrote this paper as an appendix to [**7**] back in 1988. It contains the proof of integrability of curvature of the regular part of a semi-analytic set, Proposition 1 below. This result can be proven in a simpler way using the functoriality of curvature form as for instance shown in [**1**] and that is why back in 1988 I put this appendix to a drawer. On the other hand the proof presented below is quite different than the standard one and uses techniques that can be useful, see for instance [**6**].

The proof presented in this paper follows to a big extend the ideas of the proof of a similar statement in the complex domain given by T. Mostowski in [**5**]. It is based by a direct estimate of curvature in terms of second derivatives and consequently, thanks to techniques developped in [**7**], in terms of the distances to strata of a Lipschitz stratification. Let us now outline the main points of the proof. Let $X \subset \mathbb{R}^n$ be semi-analytic and let $k = \dim X \leqslant n-1$. Decomposing X into finitely many pieces we may suppose that it is the graph of a semi-analytic mapping $\overline{U} \to \mathbb{R}^{n-k}$, with $U \subset \mathbb{R}^k$ open and semi-analytic. The integrability of the curvature forms on X reduces to the integrability on U of some combinations of the partial derivatives of F of the first and second order. The former we may suppose bounded by a more precise decomposition of X (we use the so called decomposition into L-regular sets). The second order derivatives are then bounded by the first order ones divided the distances to the strata

2000 ***Mathematics Subject Classification.*** — 32Bxx, 32C30.
Key words and phrases. — Semi-analytic sets, integrability, curvature, Lipschitz stratification.

of a stratification of $\overline{U}$. This follows from an inequality (12) that plays an important rôle in the proof of the existence of Lipschitz stratification of semi- and subanalytic sets, see Lemma 4.5 of [**7**] and Proposition 3.1 of [**9**]. Therefore the integrability of curvature is reduced to the integrability on $\overline{U}$ of functions of the form

$$A(x) = \frac{(d_s(x))^\gamma}{\prod_{j=0}^{n-1} d_j(x)},$$

where d_j denote the distances to the j dimensional strata. These functions are generally not integrable since the direct integration gives logarithms. A more delicate analysis in Lemma 4 below, shows that $A(x)$ is integrable on some "horn neighbourhoods" of strata, where the distance to a fixed stratum is dominated by the distances to the smaller strata, and as we show in Lemma 7 this is precisely what we need for the integrability of curvature. Finally Lemma 4 follows fairly easily by induction on dimension thanks to Lemmas 2 and 3 below which relate the distance to a semianalytic set and the distances to its projections and to its sections. Note that Lemmas 2-4 follows from the regular projections theorem, see [**5**], Proposition 2.1 of [**7**], and [**9**] section 5, and do not require the use of Lipschitz stratifications. In particular Lemma 4 holds for any stratification, not necessarily Lipschitz.

The paper is presented below virtually in its original form. Only the evident misprints and orthographic and gramatical errors were corrected. Since 1988 the theory of Lipschitz stratification was further developed by T. Mostowski and myself. The reader may consult [**8**] for an account of this development. In particular the regular projection theorem and the existence of Lipschitz stratification was proven for subanalytic sets [**9**], and hence all the results of this paper hold as well in the subanalytic set-up. As follows from [**9**], it is easy to bound the number of regular projections in Proposition 2.1 of [**7**]. In particular, in lemmas 2 and 3 we may take $N = n+1$ and any generic $(n+1)$-tuple of vectors $\xi_1, \dots, \xi_{n+1}$ from $\mathbb{R}^n$ satisfies the statements.

For the reader convenience, we recall briefly Dubson's argument [**1**]. Let X be a k-dimensional subanalytic subset of an n dimensional real analytic manifold M with a riemannian tensor. Let $\mathrm{G}_k(TM)$ denote the k-Grassmann bundle of TM whose fibre of $x \in M$ is the Grassmannian of k-dimensional subspaces of T_xM. We denote by T the tautological k bundle on $\mathrm{G}_k(TM)$. Note that the metric tensor on M induces a metric tensor on T. Let X_{reg} denote the regular (k-dimensional) part of X. The Nash blowing-up $\widetilde{X}$ of X is the closure in $\mathrm{G}_k(TM)$ of

$$\{(x,\xi) \in \mathrm{G}_k(TM) \mid x \in X_{\mathrm{reg}} \text{ and } \xi = T_x X_{\mathrm{reg}}\}.$$

It is known that $\widetilde{X}$ is subanalytic. Let $\pi : \widetilde{X} \to X$ denote the projection. Then, clearly, $\pi^*TX|_{X_{\mathrm{reg}}}$ coincides with $T|_{\pi^{-1}(X_{\mathrm{reg}})}$ and hence extends on $\widetilde{X}$. As a consequence the pull-back of the curvature form Ω of X_{reg} coincides, on $\pi^{-1}(X_{\mathrm{reg}})$, with the curvature form Ω_T of T. Let P be an invariant homogeneous polynomial of degree k. Then $P(\Omega)$ is integrable on each relatively compact subset Y of X_{reg}. Indeed, since π is proper $\widetilde{Y} = \pi^{-1}(Y)$ is relatively compact. Moreover, being subanalytic, $\widetilde{Y}$ has finite k-volume. On the other hand $\pi^* P(\Omega) = P(\pi^*\Omega) = p(\Omega_T)$ and the latter is integrable on $\widetilde{Y}$.

I would like to thank Tadeusz Mostowski for encourangement in preparing this paper for publication.

The aim of this paper is to prove the following proposition.

Proposition 1. — *Let M be a real analytic manifold with a given metric tensor. Let $X \subset M$ be a compact k-dimensional semi-analytic set and let Ω be the curvature form on the set X_{reg} of regular points of X of the induced metric tensor. Then, for every invariant homogeneous polynomial P of degree k, the k-form $P(\Omega)$ is integrable on X_{reg}. If X_{reg} is oriented, then* $\mathrm{Pf}(\Omega)$ *is integrable. (see, for exemple,* [**4**] *for the definition of the Pfaffian* Pf*).*

First we investigate the function of distance to a semi-analytic set. Let $X \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be a compact semi-analytic set. For a given $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi(\xi) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection parallel to $(\xi, 1)$ and by $\mathrm{dist}_\xi(x, X)$ the distance from x to X in $(\xi, 1)$ direction

$$\mathrm{dist}_\xi(x, X) := \mathrm{dist}(x, X \cap (\pi(\xi))^{-1}(\pi(\xi)(x))).$$

Of course $\mathrm{dist}_\xi(x, X) \geqslant \mathrm{dist}(x, X)$. It is a well-known fact, see [**3**], that $\mathrm{dist}(x, X)$ is a subanalytic, but not necessarily semi-analytic, function. Note that for any ξ, $\mathrm{dist}_\xi(x, X)$ is also subanalytic.

Lemma 2. — *Let X be a compact semi-analytic subset of $\mathbb{R}^n$. Then there are a finite number of vectors $\xi_1, \ldots, \xi_N \in \mathbb{R}^n$, a positive constant C, and a semi-analytic subset $Y \subset X$ such that* $\dim Y < n-1$ *and*

$$\min\{\min_j \mathrm{dist}_{\xi_j}(x, X), \mathrm{dist}(x, Y)\} \leqslant C \,\mathrm{dist}(x, X), \tag{1}$$

for all $x \in \mathbb{R}^n$.

Proof. — Since X is compact, it is sufficient to prove the lemma locally in a neighbourhood of every $x_0 \in \mathbb{R}^n$. If $x_0 \notin \mathrm{Fr}(X) = X \setminus \mathrm{Int}(X)$, putting $\xi = 0$ we obtain (1) with $C = 1$.

Let $x_0 \in \mathrm{Fr}(X)$. It suffices to prove the lemma for $\mathrm{Fr}(X)$ instead of X, so we can assume that $\dim X \leqslant n-1$. We complexify $\mathbb{R}^n$ and consider a complex hypersurface $\widetilde{X}$ in an open neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X \cap \widetilde{U} \subset \widetilde{X}$. Take constants $C, \varepsilon > 0$ and vectors $\xi_1, \ldots, \xi_N$ satisfying the assertion of Corollary 2.4 of [**7**] for $(\widetilde{X}, x_0)$. In particular, for every x close to x_0 there exists $\xi \in \{\xi_1, \ldots, \xi_N\}$ such that the intersection of the open cone

$$S_\varepsilon(x,\xi) = \{x + \lambda(\eta, 1) \mid |\eta - \xi| < \varepsilon, \lambda \in \mathbb{C}^*\}$$

with X is of the form given in (8) of [**7**]. We recall for the reader's convenience that it means that

$$S_\varepsilon(x,\xi) \cap X = \bigcup_i \{x + \lambda_i(\eta)(\eta, 1) \mid |\eta - \xi| < \varepsilon\},$$

where λ_i, $i = 1, \dots, r$, are real analytic functions defined on $|\eta - \xi| < \varepsilon$ and satisfying $\lambda_i(\eta) \neq \lambda_j(\eta)$ for $i \neq j$ and all η, and $|D\lambda_i| \leqslant C|\lambda_i|$. Furthermore we may assume that for each j, $\pi(\xi_j)|_X$ is a branched analytic covering and let $B(\xi_j)$ be its critical locus. Put $Y = \bigcup_j \pi(\xi_j)^{-1}(B(\xi_j)) \cap X$. Clearly Y is semi-analytic and $\dim Y < n-1$.

Let U be a sufficiently small neighbourhood of x_0 such that $U \subset \widetilde{U} \cap \mathbb{R}^n$. Let $x \in U$ and we assume that the regular projection corresponding to x is standard $\mathbb{R}^n \to \mathbb{R}^{n-1}$. Let $p \in X$ be one of the points nearest to x. Let $p' = \pi(p)$, $x' = \pi(x)$, and let $U' = \pi(U)$. If $x' = p'$ then $\operatorname{dist}(x, X) = \operatorname{dist}_\xi(x, X)$. So, we assume $x' \neq p'$ and consider the segment $\overline{p'x'}$. Starting from p we lift $\overline{p'x'}$ to a smooth curve γ on X until we reach a point $s \in Y$ or s of the form $(x', \lambda_i(0))$ for some $i = 1, \dots, r$. We denote $\pi(s)$ by s'. It remains to prove that

$$|x - s| \leqslant C|x - p|, \tag{2}$$

for a universal constant C. This follows from Remark 2.5 of [**7**]. More precisely, if $p \in S_{\varepsilon'}(x, 0)$, where ε' is given by Remark 2.5 of [**7**], the length of γ is estimated by $C'|p' - s'|$. Hence

$$|x - s| \leqslant |x - p| + |p - s| \leqslant |x - p| + C'|p' - s'| \leqslant |x - p| + C'|p' - x'|$$

and consequently (2) follows. If $s \notin S_{\varepsilon'}(x, 0)$ then

$$|x - s| \leqslant C|x' - s'| \leqslant C|x' - p'| \leqslant C|x - p|.$$

If $s \in S_{\varepsilon'}(x, 0)$ and $p \notin S_{\varepsilon'}(x, 0)$, then we may find $r \in \gamma \cap \operatorname{Fr}(S_{\varepsilon'}(x, 0))$ and by the above

$$|x - s| \leqslant |x - r| + |r - s| \leqslant C'|x' - r'| \leqslant C'|x' - p'| \leqslant C|x - p|. \qquad \square$$

Lemma 3. — *Let X be a semi-analytic subset of $\mathbb{R}^n$, $\dim X < n-1$, and $x_0 \in \mathbb{R}^n$. Then there exist a finite number of vectors $\xi_1, \dots, \xi_N \in \mathbb{R}^n$ and constants $C, \varepsilon > 0$ such that for a sufficiently small neighbourhood U of x_0 and every $x \in U$ there is ξ_j such that $X \cap U \subset \mathbb{R}^n \smallsetminus S_\varepsilon(x, \xi_j)$. In particular*

$$\operatorname{dist}(x, X) \leqslant C \max_j \{\operatorname{dist}(\pi(\xi_j)(x), \pi(\xi_j)(X \cap U))\}.$$

Proof. — It is sufficient to prove the lemma for $x_0 \in X$. Complexify $\mathbb{R}^n$ and consider complex hypersurfaces $\widetilde{X}_1, \widetilde{X}_2$ in an open neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X \cap \widetilde{U} \subset \widetilde{X}_1 \cap \widetilde{X}_2$ and $\dim_{\mathbb{C}} \widetilde{X}_1 \cap \widetilde{X}_2 = n-2$. Then the lemma follows from Proposition 2.1 of [**7**] applied to $\widetilde{X}_1 \cup \widetilde{X}_2$. $\square$

Now we consider the following situation. Let X be a compact semi-analytic subset of $\mathbb{R}^n$ and $\dim X = n$. Let

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n = X$$

be a family of semi-analytic subsets of X such that $\dim X^i \leqslant i$ for each i. For any $N \in \mathbb{N}$, $C > 0$ and $j = 0, \ldots, n-1$, consider the following subsets of $U = X \setminus (\mathrm{Fr}(X) \cup X^{n-1})$

$$U_{N,C,j} = \{x \in U \mid d_j(x) < C[d_{j-1}(x)]^N\},$$

where $d_j(x) = \mathrm{dist}(x, X^j)$. (If $X_j = \varnothing$ then we mean $d_j \equiv 1$).

Lemma 4. — *For any $N, N' \geqslant 1$, $C, C' > 0$, $\gamma > 0$, $s = 0, \ldots, n-1$, the function*

$$A(x) = \frac{(d_s(x))^\gamma}{\prod_{j=0}^{n-1} d_j(x)}$$

is integrable on $U_{N,C,s} \setminus \bigcup_{j>s} U_{N',C',j}$.

Proof. — Induction on $n = \dim X$.

Since X is compact, it suffices to prove the lemma locally, that is in a neighbourhood of each point of X. Fix $x_0 \in X$. Assume that X is contained in a sufficiently small neighbourhood V of x_0. We apply Lemma 2 to X^{n-1} and Lemma 3 to X^{n-2} at x_0. We can do it simultaneously and obtain a finite number $\xi_1, \ldots, \xi_N$ of vectors and a semi-analytic subset Y of X, $\dim Y < n-1$, such that for every $x \in V$ (shrinking V if necessary), either for some ξ_j

$$\mathrm{dist}_{\xi_j}(x, X^{n-1}) \leqslant C d_{n-1}(x) \tag{3}$$

$$\text{and}\quad d_r(x) \leqslant C\,\mathrm{dist}(\pi(\xi_j)(x), \pi(\xi_j)(X^r)), \tag{4}$$

for each $r = 0, 1, \ldots, n-2$, or

$$\mathrm{dist}(x, Y) \leqslant C d_{n-1}(x), \tag{5}$$

for some $C > 0$. Indeed, we can find complex hypersurfaces $\widetilde{X}_1, \widetilde{X}_2$ of a neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X^{n-1} \cap \widetilde{U} \subset \widetilde{X}_1$, $X^{n-2} \cap \widetilde{U} \subset \widetilde{X}_1 \cap \widetilde{X}_2$, $\dim_{\mathbb{C}}(\widetilde{X}_1 \cap \widetilde{X}_2) < n-1$. Then $\xi_1, \ldots, \xi_N$ given by Corollary 2.4 of [**7**] applied to $(\widetilde{X}_1 \cup \widetilde{X}_2, x_0)$ satisfies the properties claimed above (see also the proofs of Lemmas 2 and 3).

Apply again Lemma 3 to $Y \cup X^{n-2}$ at x_0 and add the obtained vectors to the set $\xi_1, \ldots, \xi_N$. In conclusion, for each $x \in V$ there is ξ_j so that the inequality (4) holds for $r = 0, \ldots, n-2$ and

$$\mathrm{dist}(x, Y) \leqslant C\,\mathrm{dist}(\pi(\xi_j)(x), \pi(\xi_j)(Y)) \tag{6}$$

$$S_\varepsilon(x, \xi_j) \cap Y = \varnothing. \tag{7}$$

Furthermore, we may require that for each ξ_j, $\pi(\xi_j))|_{X^{n-1}}$ is finite and $\pi(\xi_j)(Y)$, $\pi(\xi_j)(X^r)$, $r = 0, \ldots, n$, are semi-analytic subsets of $\mathbb{R}^{n-1}$ (see [**2**]).