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COMPUTATIONAL ASPECTS OF GROTHENDIECK LOCAL RESIDUES

by

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Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

Abstract. — Grothendieck local residues are studied from a view point of algebraic analysis. The main idea in this approach is the use of regular holonomic $\mathcal{D}$-modules attached to a zero-dimensional algebraic local cohomology class. A new method for computing Grothendieck local residues is developed in the context of Weyl algebra. An effective computing algorithm that exploits first order annihilators is also described.

***Résumé* (Aspects effectifs des résidus locaux de Grothendieck).** — On étudie le résidu local de Grothendieck du point de vue de l'analyse algébrique. L'idée principale de cette approche est l'utilisation de $\mathcal{D}$-modules holonomes réguliers attachés à une classe algébrique de cohomologie locale en dimension zéro. On développe une méthode nouvelle pour calculer les résidus locaux de Grothendieck dans le cadre de l'algèbre de Weyl. Cette méthode permet de décrire un algorithme efficace, lequel utilise les annulateurs du premier ordre.

1. Introduction

In this paper, we consider Grothendieck local residues and its duality in the context of holonomic $\mathcal{D}$-modules. Upon using the regular holonomic system associated to a certain zero-dimensional algebraic local cohomology class, we derive a method for computing Grothendieck local residues. We also give an effective algorithm that serves exact computations.

In § 2, we study local residues from the viewpoint of the analytic $\mathcal{D}$-module theory. By using the local residue pairing, we associate to an algebraic local cohomology class attached to a given regular sequence an analytic linear functional acting on the space of germs of holomorphic functions. We apply Kashiwara-Kawai duality theorem on holonomic systems [**3**] to the residue pairing and show that the kernel of the above analytic functional can be described in terms of partial differential operators. This result ensures in particular the computability of the Grothendieck local residues.

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In § 3, we give a framework in the Weyl algebra, and develop there a method for computing Grothendieck local residues. The key ingredient of the present method is the annihilating ideal in the Weyl algebra of the given zero-dimensional algebraic local cohomology class. We show that the use of generators of the annihilating ideal in the Weyl algebra reduces the computation of the local residues to that of linear equations.

In § 4, we derive an algorithm for computing Grothendieck local residues that exploits only first order partial differential operators. The resulting algorithm (Algorithm R) is efficient and thus can be available in use for actual computations in many cases. We also present an criterion to the applicability of this algorithm.

In § 5, we give an example to illustrate an effectual way of using Algorithm R.

In Appendix, we present an algorithm that outputs the first order partial differential operators which annihilate a direct summand in question of the given algebraic local cohomology class.

2. Local duality theorem

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X = \mathbb{C}^n$ and $\mathcal{F}$ a regular sequence given by n holomorphic functions $f_1, \ldots, f_n$ on X. Denote by $\mathcal{I}$ the ideal of $\mathcal{O}_X$ generated by $f_1, \ldots, f_n$ and Z the zero-dimensional variety

$$\boldsymbol{V}(\mathcal{I}) = \{z \in X \mid f_1(z) = \cdots = f_n(z) = 0\}$$

of the ideal $\mathcal{I}$ consisting of finitely many points.

There is a canonical mapping ι from the sheaf of n-th extension groups $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X)$ to the sheaf of n-th algebraic local cohomology groups $\mathcal{H}^n_{[Z]}(\Omega^n_X)$ with support on Z:

$$\iota : \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X) \longrightarrow \mathcal{H}^n_{[Z]}(\Omega^n_X)$$

where Ω^n_X is the sheaf of holomorphic n-forms on X. We denote by $\omega_{\mathcal{F}} = \left[\dfrac{dz}{f_1 \cdots f_n}\right]$ the image by the mapping ι of the Grothendieck symbol

$$\begin{bmatrix} dz \\ f_1 \ldots f_n \end{bmatrix} \in \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \Omega^n_X),$$

i.e.,

$$\omega_{\mathcal{F}} = \iota\left(\begin{bmatrix} dz \\ f_1 \ldots f_n \end{bmatrix}\right) \in \mathcal{H}^n_{[Z]}(\Omega^n_X), \tag{1}$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$. Let $\omega_{\mathcal{F},\beta}$ denote the germ at $\beta \in Z$ of the algebraic local cohomology class $\omega_{\mathcal{F}}$:

$$\omega_{\mathcal{F},\beta} \in \mathcal{H}^n_{[\beta]}(\Omega^n_X),$$

where $\mathcal{H}^n_{[\beta]}(\Omega^n_X)$ stands for the algebraic local cohomology supported at β.

Let $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ be the sheaf of n-th local cohomology groups at $\beta \in Z$ and let $\mathrm{Res}_\beta : \mathcal{H}^n_{\{\beta\}}(\Omega^n_X) \to \mathbb{C}$ be the local residue map. Recall that the mapping

$$\mathcal{H}^n_{\{\beta\}}(\Omega^n_X) \times \mathcal{O}_{X,\beta} \longrightarrow \mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$$

composed with the local residue map Res_β defines a natural pairing between two topological vector spaces $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ and $\mathcal{O}_{X,\beta}$. Thus, the algebraic local cohomology class $\omega_{\mathcal{F},\beta} \in \mathcal{H}^n_{[\beta]}(\Omega^n_X)$ which also belongs to $\mathcal{H}^n_{\{\beta\}}(\Omega^n_X)$ induces a linear functional $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ that acts on $\mathcal{O}_{X,\beta}$. Namely, $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ is defined to be

$$\mathrm{Res}_\beta(\omega_{\mathcal{F}})(\varphi(z)) = \mathrm{Res}_\beta(\varphi(z)\omega_{\mathcal{F},\beta})$$

for $\varphi(z) \in \mathcal{O}_{X,\beta}$, $\beta \in Z$. We consider the kernel space Ker of the linear functional $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$ defined to be

$$\mathrm{Ker} = \{\psi(z) \in \mathcal{O}_{X,\beta} \mid \mathrm{Res}_\beta(\omega_{\mathcal{F}})(\psi(z)) = 0\}.$$

Now we are going to give an alternative description of the kernel space Ker in terms of partial differential operators.

Let $\mathcal{D}_X$ be the sheaf on X of linear partial differential operators. Then the sheaves Ω^n_X, $\mathcal{H}^n_{[\beta]}(\Omega^n_X)$ and $\mathcal{H}^n_{[Z]}(\Omega^n_X)$ are *right* $\mathcal{D}_X$-modules. Note also that $\mathcal{O}_X$ and $\mathcal{H}^n_{[\beta]}(\mathcal{O}_X)$ have a structure of left $\mathcal{D}_X$-modules. We denote by $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ the *right* ideal of $\mathcal{D}_X$ consisting of linear partial differential operators which annihilate the cohomology class $\omega_{\mathcal{F}}$:

$$\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}) = \{P \in \mathcal{D}_X \mid \omega_{\mathcal{F}}P = 0\}.$$

Note that, if we set $\omega_{\mathcal{F}} = \sigma_{\mathcal{F}}dz$ with $\sigma_{\mathcal{F}} \in \mathcal{H}^n_{[Z]}(\mathcal{O}_X)$, the right ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ can be rewritten as

$$\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}) = \{P \in \mathcal{D}_X \mid P^*\sigma_{\mathcal{F}} = 0\},$$

where P^* stands for the formal adjoint operator of P.

The $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is isomorphic to $\mathcal{H}^n_{[Z]}(\Omega^n_X)$. We thus in particular have the following theorem (*cf.* [**2**], [**3**], [**7**]);

Theorem 2.1. — *Let $\mathcal{F}$ be a regular sequence given by n holomorphic functions and $\omega_{\mathcal{F}}$ an algebraic local cohomology class defined by* (1) *whose support contains a point β.*

(i) *$\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is a regular singular holonomic system.*
(ii) *$\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is simple at each point $\beta \in Z$.*

The theorem implies the folloiwng result on the local cohomology solution space of the holonomic system $\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$;

Corollary 2.2. — *Let $\beta \in Z$. Then*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}), \mathcal{H}^n_{\{\beta\}}(\Omega^n_X)) &= \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}}), \mathcal{H}^n_{[\beta]}(\Omega^n_X)) \\ &= \mathbb{C}\omega_{\mathcal{F},\beta}\end{aligned}$$

holds.

The above result means that the holonomic system $\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ completely characterize the algebraic local cohomology class $\omega_{\mathcal{F}}$ as its solution.

***Example 2.3* (*cf.* [1]).** — Let $\mathcal{F}=\{f_1,f_2\}$ be a regular sequence and I be the ideal in $\mathbb{C}[x,y]$ generated by functions f_1 and f_2 given below. Let $j_{\mathcal{F}}(x,y)=\det\left(\dfrac{\partial(f_1,f_2)}{\partial(x,y)}\right)$ be the Jacobian of f_1 and f_2. We fix the lexicographical ordering $x\succ y$ and use the term ordering $\succ$ in computations of Gröbner basis of I.

(i) Let $f_1=x(x^2-y^3-y^4)$, $f_2=x^2-y^3$. We have $I=\langle x^2-y^3,xy^4,y^7\rangle$ and $\boldsymbol{V}(I)=\{(0,0)\}$ with the multiplicity 11. The algebraic local cohomology class $\omega_{\mathcal{F}}=\left[\frac{dx\wedge dy}{f_1f_2}\right]$ is supported only at the origin $(0,0)$. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of $\omega_{\mathcal{F}}$ is generated by multiplication operators $x(x^2-y^3-y^4)$, x^2-y^3 and a first order differential operator $P=3x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}-12$. By solving the system of differential equations $\omega_{\mathcal{F}}y^7=\omega_{\mathcal{F}}xy^4=\omega_{\mathcal{F}}(x^2-y^3)=\omega_{\mathcal{F}}P=0$ together with the formula $j_{\mathcal{F}}(x,y)\omega_{\mathcal{F}}=11\delta_{(0,0)}dx\wedge dy$ where $\delta_{(0.0)}=\left[\frac{1}{xy}\right]\in\mathcal{H}^2_{[(0,0)]}(\mathcal{O}_X)$ is the delta function with support at the origin, we have the following representation of $\omega_{\mathcal{F}}$;

$$\omega_{\mathcal{F}}=\left[\left(\frac{1}{x^5y}+\frac{1}{x^3y^4}+\frac{1}{xy^7}\right)dx\wedge dy\right].$$

(ii) Let $f_1=x$ and $f_2=(x^2-y^3)(x^2-y^3-y^4)$. We have $I=\langle x,y^7+y^6\rangle$ and its primary decomposition $I=\langle x,y+1\rangle\cap\langle x,y^6\rangle$. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of the algebraic local cohomology class $\omega_{\mathcal{F}}=\left[\frac{dx\wedge dy}{f_1f_2}\right]$ is generated by x,y^7+y^6 and $P=(y^2+y)\frac{\partial}{\partial y}-5y-5$. We have a representation

$$\left[\left(\frac{1}{xy}-\frac{1}{xy^2}+\frac{1}{xy^3}-\frac{1}{xy^4}+\frac{1}{xy^5}-\frac{1}{xy^6}\right)dx\wedge dy\right]+\left[\frac{dx\wedge dy}{x(y+1)}\right]$$

of $\omega_{\mathcal{F}}$ by solving the system of differential equations $\omega_{\mathcal{F}}x=\omega_{\mathcal{F}}(y^6+y^7)=\omega_{\mathcal{F}}P=0$ together with the formula $j_{\mathcal{F}}(x,y)\omega_{\mathcal{F}}=(6\delta_{(0,0)}+\delta_{(0,-1)})dx\wedge dy$ where $\delta_{(0,-1)}=\left[\frac{1}{x(y+1)}\right]$ is the delta function with support at $(0,-1)$.

(iii) Let $f_1=x^2-y^3-y^4$ and $f_2=x(x^2-y^3)$. We have $I=\langle x^2-y^4-y^3,xy^4,y^8+y^7\rangle$ and its primary decomposition $I=\langle x,y+1\rangle\cap\langle x^2-y^4-y^3,xy^4,y^7\rangle$. The variety $\{(0,-1)\}$ is simple and $\{(0,0)\}$ is of multiplicity 11. The annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ of the algebraic local cohomology class $\omega_{\mathcal{F}}=\left[\frac{dx\wedge dy}{f_1f_2}\right]$ is generated by $x^2-y^4-y^3,xy^4,y^8+y^7$ and $P=(4xy+3x)\frac{\partial}{\partial x}+(2y^2+2y)\frac{\partial}{\partial y}-12y-12$. We have a representation of $\omega_{\mathcal{F}}$ as

$$\left[\left(\frac{1}{xy}+\frac{1}{x^5y}-\frac{1}{xy^2}+\frac{1}{xy^3}-\frac{1}{xy^4}+\frac{1}{x^3y^4}+\frac{1}{xy^5}-\frac{1}{xy^6}+\frac{1}{xy^7}\right)dx\wedge dy\right]+\left[-\frac{dx\wedge dy}{x(y+1)}\right]$$

by solving $\omega_{\mathcal{F}}(x^2-y^3-y^4)=\omega_{\mathcal{F}}xy^4=\omega_{\mathcal{F}}(y^8+y^7)=\omega_{\mathcal{F}}P=0$ together with the formula $j_{\mathcal{F}}(x,y)\omega_{\mathcal{F}}=(11\delta_{(0,0)}+\delta_{(0,-1)})dx\wedge dy$.

***Example 2.4* ([4])**. — Let $f=x^3+y^7+xy^5$. We consider the regular sequence given by partial derivarives $f_1=3x^2+y^5$ and $f_2=5xy^4+7y^6$ of f. The primary decomposition of the ideal $I=\langle f_1,f_2\rangle$ is given by $\langle 3125x+151263, 25y+147\rangle\cap I_0$ where $I_0=\langle 3x^2+y^5, 5xy^4+7y^6, y^8\rangle$.

For a direct summand ω_1 with support at $\{(-\frac{151263}{3125},-\frac{147}{25})\}$of the algebraic local cohomology class $\omega_{\mathcal{F}}=\left[\frac{dx\wedge dy}{f_1f_2}\right]$, the annihilating ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_1)$ is given by $\langle 25y+147, 3125x+151263\rangle\mathcal{D}_X$.

For the other direct summand ω_0 with support at the origin $(0,0)$, its annihilating ideal $\mathrm{Ann}_{A_n}(\omega_0)$ is generated by the ideal I_0 and the second order differential operator

$$\begin{aligned}
&y\frac{\partial^2}{\partial y^2}+\Big(-\frac{43}{18}y^4+\frac{84}{5}xy\Big)\frac{\partial^2}{\partial x^2}+\Big(\frac{50}{147}y+9\Big)\frac{\partial}{\partial y}\\
&+\bigg(\frac{6250}{1361367}y^4+\frac{125}{9261}y^3+\Big(-\frac{78125}{3176523}x-\frac{5}{63}\Big)y^2+\Big(\frac{8125}{64827}x+\frac{252}{5}\Big)y-\frac{25}{441}x\bigg)\frac{\partial}{\partial x}\\
&\quad-\frac{762939453125}{218041257467152161}y^7+\frac{6103515625}{494424620106921}y^6-\frac{8300781250}{30270895108587}y^5\\
&\quad+\frac{156250000}{205924456521}y^4+\Big(-\frac{37841796875}{211896265760109}x+\frac{781250}{1400846643}\Big)y^3\\
&\quad+\Big(\frac{927734375}{1441471195647}x-\frac{78125}{1361367}\Big)y^2+\Big(-\frac{1953125}{1400846643}x+\frac{21250}{64827}\Big)y\\
&\quad-\frac{390625}{66706983}x+\frac{650}{441}.
\end{aligned}$$

Kashiwara-Kawai duality theory on holonomic systems ([**3**]) together with Theorem 2.1 implies the following result which gives a characterization of the space Ker.

Theorem 2.5. — *Let* Ker *be the kernel space of the residue mapping* $\mathrm{Res}_\beta(\omega_{\mathcal{F}})$. *Then*

$$\mathrm{Ker}=\{R\varphi(z)\mid \varphi(z)\in\mathcal{O}_{X,\beta}, R\in\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})\}$$

holds.

Observe that the stalk at $\beta\in Z$ of $\mathcal{O}_X/\mathcal{I}$ is a finite dimensional vector space, the quotient space $\mathrm{Ker}/\mathcal{I}\subset\mathcal{O}_X/\mathcal{I}$ is a one codimensional vector subspace. Hence, if generators of the ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ are given, the determination of Ker can be reduced to a problem in the finite dimensional vector space.

Example 2.6. — Let $f_1=x^3$ and $f_2=y^2+2x^2+3x$. The variety $\boldsymbol{V}(I)$ of the ideal $I=\langle f_1,f_2\rangle$ is the origin $\{(0,0)\}$ with the multiplicity 6. Let $\omega_{\mathcal{F}}=\left[\frac{dx\wedge dy}{f_1f_2}\right]\in\mathcal{H}^2_{[(0,0)]}(\Omega^n_X)$. Then the right ideal $\mathcal{A}nn_{\mathcal{D}_X}(\omega_{\mathcal{F}})$ is generated by f_1, f_2 and the first order differential operator

$$P=6x\frac{\partial}{\partial x}+(3y+2xy)\frac{\partial}{\partial y}+(-2x-15).$$