*Séminaires & Congrès*
**10**, 2005, p. 323–340

# ON SOME CLASSES OF WEAKLY KODAIRA SINGULARITIES

*by*

Tadashi Tomaru

***Abstract.*** — In this paper, we prove some relations between surface singularities and pencils of compact complex algebraic curves. Let $(X, o)$ be a complex normal surface singularity. Let $p_f(X, o)$ be the arithmetic genus of the fundamental cycle associated to $(X, o)$. If there is a pencil of curves of genus $p_f(X, o)$ (*i.e.*, $\Phi\colon S \to \Delta$, where $\Phi$ is a proper holomorphic map between a non-singular complex surface and a small open disc in $\mathbb{C}^1$ around the origin $\{0\}$ and the fiber $S_t = \Phi^{-1}(t)$ is a smooth compact algebraic curve of genus $p_f(X, o)$ for any $t \neq 0$) and a resolution $(\widetilde{X}, E) \to (X, o)$ such that $(S, \mathrm{supp}(S_o)) \supset (\widetilde{X}, E)$, then we call $(X, o)$ *a weakly Kodaira singularity.* Any Kodaira singularity in the sense of Karras is a weakly Kodaira singularity. In this paper we show some sufficient conditions for surface singularities of some classes to be weakly Kodaira singularities.

***Résumé* (Sur certaines classes de singularités faiblement Kodaira).** — Dans cet article, nous montrons certaines relations entre les singularités de surfaces et les pinceaux de courbes algébriques complexes compactes. Soit $(X, o)$ une singularité de surface complexe normale. Soit $p_f(X, o)$ le genre arithmétique du cycle fondamental associé à $(X, o)$. S'il existe un pinceau de courbes de genre $p_f(X, o)$ (*i.e.*, s'il existe une application holomorphe propre $\Phi\colon S \to \Delta$, entre une surface complexe non-singulière et un petit disque ouvert dans $\mathbb{C}^1$ autour de l'origine $\{0\}$ tels que la fibre $S_t = \Phi^{-1}(t)$ soit une courbe algébrique lisse compacte de genre $p_f(X, o)$ pour tout $t \neq 0$) et une résolution $(\widetilde{X}, E) \to (X, o)$ telle que $(S, \mathrm{supp}(S_o)) \supset (\widetilde{X}, E)$, alors on dit que $(X, o)$ est *une singularité faiblement Kodaira*. Toute singularité Kodaira dans le sens de Karras est une singularité faiblement Kodaira. Dans cet article, nous montrons certaines conditions suffisantes pour que les singularités de surface de certaines classes soient des singularités faiblement Kodaira.

## 1. Introduction

After Kulikov's work ([**4**]) on Arnold's uni- and bi-modal singularities, U. Karras ([**3**]) introduced the notion of Kodaira singularities, which was defined by pencils

**2000 *Mathematics Subject Classification.*** — Primary 32S10, 32S25; Secondary 14D06.
***Key words and phrases.*** — Normal surface singularity, pencil genus, pencil of curves, weakly Kodaira singularity.

of curves (*i.e.*, one parameter families of compact complex algebraic curves). Also, J. Stevens [**8**] studied a subclass of Kodaira singularities (called *Kulikov singularities*). They applied them to deformation theory of singularities. In this paper, we also consider normal surface singularities associated to pencils of curves (*i.e.*, weakly Kodaira singularities).

In [**13**], the author introduced an invariant for normal surface singularities, which is associated to pencils of curves, and proved some results. We explain the definition. Let $S$ be a complex surface and $\Delta$ a small open disk in the complex line $\mathbb{C}^1$ around the origin. A holomorphic mapping $\Phi\colon S \to \Delta$ is called *a pencil of curves of genus $g$* if $\Phi$ is proper and surjective and the fiber $S_t = \Phi^{-1}(t)$ is a smooth compact complex curve of genus $g$ for any $t$ with $t \neq 0$. Let $(X, o)$ be a normal surface singularity. We consider the following property:

(1.1) There exists a good resolution $\pi\colon (\widetilde{X}, E) \to (X, o)$ and a pencil of curves $\Phi\colon S \to \Delta$ such that $(S, \mathrm{supp}(S_o)) \supset (\widetilde{X}, E)$ (*i.e.*, $S \supset \widetilde{X}$ and $\mathrm{supp}(S_o) \supset E$).

***Definition 1.1***

(i) Let us define

$$p_e(X, o) := \min\{\text{the genus of a pencil of curves satisfying (1.1)}\},$$

and call it *the pencil genus of* $(X, o)$.

(ii) Let $h$ be an element of $\mathfrak{m}_{X,o}$ such that the divisor of $\mathrm{red}(h \circ \pi)_{\widetilde{X}}$ is simple normal crossing. Consider pencils of curves $\Phi\colon S \to \Delta$ satisfying (1.1) and $h \circ \pi = \Phi$. Let us define

$$p_e(X, o, h) := \min\{\text{genus of such a pencil of curves}\},$$

and call it $p_e(X, o, h)$ *the pencil genus of a pair of* $(X, o)$ *and* $h$.

For $\widetilde{X}$ and $h$ as above, the author constructed a pencil of curves of genus $p_e(X, o, h)$ that satisfy (1.1) and $h \circ \pi = \Phi$ ([**13**], Theorem 2.2). The surface $S$ of Definition 1.1 is constructed by glueing $\widetilde{X}$ and suitable resolution spaces of some cyclic quotient singularities. In [**13**], he also proved some results for $p_e(X, o)$ and $p_e(X, o, h)$. For example, Kodaira and Kulikov singularities are characterized by using them. Moreover, the author [**13**] proved an estimate of (1.2) on $p_e(X, o)$. Let $(X, o)$ be a normal surface singularity and $\sigma\colon (\widetilde{X}, E) \to (X, o)$ a resolution and $Z_E$ the fundamental cycle on $E$. Since the arithmetic genus $p_a(Z_E)$ of $Z_E$ is independent of the choice of a resolution, $p_a(Z_E)$ is an invariant of $(X, o)$ ([**14**]). Then we define it as $p_f(X, o)$ and call it the *fundamental genus* of $(X, o)$. Also, $p_f(X, o)$ is a topological invariant of $(X, o)$ and it is useful for a rough classification of normal surface singularities. In [**13**], the author proved that

$$p_f(X, o) \leqslant p_e(X, o) \leqslant p_a(\mathbb{M}_X) + \mathrm{mult}(X, o) - 1, \tag{1.2}$$

where $\mathrm{mult}(X,o)$ is the multiplicity of $(X,o)$ and $\mathbb{M}_X$ is the maximal ideal cycle on the minimal resolution of $(X,o)$. From Karras's result [**3**], if $(X,o)$ is a Kodaira singularity, we have $p_e(X,o) = p_f(X,o)$. Therefore we give the following definition.

***Definition 1.2***. — If $p_f(X,o) = p_e(X,o) = g$, then we call $(X,o)$ *a weakly Kodaira singularity of genus* $g$.

Though any Kodaira singularity is a weakly Kodaira singularity, the converse is not necessarily true. For rational double points, every $A_n$-singularity is a Kodaira singularity and every $D_n$-singularity ($n \geqslant 4$) is a weakly Kodaira singularity but not a Kodaira singularity. Since rational double points of $E_6$, $E_7$ and $E_8$ have $p_e(X,o) = 1$ ([**13**]), they are not weakly Kodaira singularities.

In this paper, we give some conditions to be weakly Kodaira singularities for normal surface singularities. In section 2, we consider normal surface singularities obtained through some procedures for pencils of curves, and prove a sufficient condition for them to be weakly Kodaira singularities. From this results, we can see that the class of weakly Kodaira singularities is fairly bigger than the class of Kodaira singularities. Also we prove some results on elliptic (*i.e.*, $p_f(X,o) = 1$) weakly Kodaira singularities. In section 3, we prove a sufficient condition for some cyclic coverings of normal surface singularities to be weakly Kodaira singularities. As a corollary, we obtain a class of weakly Kodaira hypersurface singularities which contains rational double points of $D_n$-type.

**Notation and terminology.** — Let $M$ be a complex surface and $E = \bigcup_{j=1}^{r} E_j \subset M$ a 1-dimensional compact analytic subspace, where $E_1, \dots, E_r$ are all irreducible components of $E$. Suppose that $E = \sum_{j=1}^{r} E_j$ is a simple normal crossing divisor on $M$ with $E_i^2 \leqslant 0$. For $(M,E)$, the *weighted dual graph* (=w.d.graph) $\Gamma_E$ of $E$ is a graph such that each vertex of $\Gamma_E$ represents an irreducible component $E_j$ weighted by $E_j^2$ and $g(E_j)$ (=genus), while each edge connecting to $E_i$ and $E_j$, $i \neq j$, corresponds to the point $E_i \cap E_j$. For example, if $E_i^2 = -b_i$ and $g(E_i) = g_i > 0$ (resp. $g_i = 0$), then $E_i$ corresponds to a vertex which is figured as follows:

$-b_i$ $[g_i]$ (resp. $-b_i$ ), and ○ means $-2$ .

Moreover, if $D = \sum_{i=1}^{r} d_i E_i$ is a cycle on $E$, then we denote by $\mathrm{Coeff}_{E_i} D$ the coefficient $d_i$. If $E_i$ is a $\mathbb{P}^1$ (*i.e.*, non-singular rational curve) with $E_i^2 = -1$, then we call it a $(-1)$-curve. If $E_i$ is a $(-1)$-curve in $E$ which intersects with only one component of $E$, we call it a $(-1)$-*edge curve* of $E$. For a resolution $\pi\colon (\widetilde{X}, E) \to (X,o)$ and an element $h \in \mathcal{O}_{X,o}$, let $(h \circ \pi)_{\widetilde{X}}$ be the divisor defined by $h \circ \pi$ on $\widetilde{X}$. Also let $E(h \circ \pi)$ (resp. $\Delta(h \circ \pi)$) be the exceptional part (resp. the non-exceptional part) of $(h \circ \pi)_{\widetilde{X}}$. Namely, we have $E(h \circ \pi) = \sum_{i=1}^{r} v_{E_i}(h \circ \pi) E_i$ and $\Delta(h \circ \pi) = \sum_{j=1}^{s} v_{C_j}(h \circ \pi) C_j$ if $\mathrm{supp}(\Delta(h \circ \pi)) = \bigcup_{j=1}^{s} C_j$, and so $(h \circ \pi)_{\widetilde{X}} = E(h \circ \pi) + \Delta(h \circ \pi)$. For any real number $a \in \mathbb{R}$, we denote by $\{a\}$ the least number greater than, or equal to $a$.

## 2. Weakly Kodaira singularities obtained by Kulikov process for pencils of curves

In this section we consider a procedure to obtain normal surface singularities from pencils of curves (originally introduced by Kulikov [**4**]). We give conditions for such singularities to be weakly Kodaira singularities. Also we prove a formula of the geometric genus when such singularities are elliptic.

Let $E$ be the exceptional set of a resolution of a normal surface singularity or $\mathrm{supp}(S_o)$ for a pencil of curves $\Phi\colon S \to \Delta$. Let $F = \bigcup_{i=1}^{r} F_i$ and $A$ be two 1-dimensional analytic subsets of $E$ such that $F_i \not\subset A$ for $i = 1, \ldots, r$. Let us consider the following three conditions:

(i) $F_i \simeq \mathbb{P}^1$ and $A \cdot F_1 = F_1 \cdot F_2 = \cdots = F_{r-1} \cdot F_r = 1$,

(ii) $F$ intersects $A$ only at $F_1 \cap A$,

(iii) $\bigcup_{i=2}^{r} F_i$ does not contain any $(-1)$ curve.

If $F$ satisfies (i) and (ii), then we call it a $\mathbb{P}^1$*-chain (of length $r$) started from $A$*. If $b_i = -F_i^2$ for any $i$, then we call it *a $\mathbb{P}^1$-chain of type $(b_1, \ldots, b_r)$ started from $A$*. If a $\mathbb{P}^1$-chain $F$ satisfies (iii), then we call it a *minimal $\mathbb{P}^1$-chain started from $A$*.

Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a pencil of curves and let $\overline{S}_o = \overline{\Phi}^{-1}(o) = \sum_{j=1}^{r} a_j A_j$ be the singular fiber. If $\gcd(a_1, \ldots, a_r) > 1$ (resp. $= 1$), then we say that the pencil is multiple (resp. non-multiple).

***Definition 2.1***

(i) Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a non-multiple pencil of curves without any $(-1)$-edge curve. Let $S^{(0)} = \overline{S} \xleftarrow{\sigma_1} S^{(1)}$ be blow-ups at non-singular points $P_1^{(1)}, \ldots, P_{t_1}^{(1)}$ of $\mathrm{red}(S_o^{(0)})$. As next step, let $P_1^{(2)}, \ldots, P_{t_2}^{(2)} \in \bigcup_{j=1}^{t_1} \sigma_1^{-1}(P_j^{(1)})$ be non-singular points of $\mathrm{red}(S_o^{(1)})$ and let $S^{(1)} \xleftarrow{\sigma_2} S^{(2)}$ be blow-ups at these points. After continuing this process $m$ times, we get $S^{(0)} = \overline{S} \xleftarrow{\sigma_1} S^{(1)} \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_m} S^{(m)} = S$ and put $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$. Hence we get a new pencil $\Phi = \overline{\Phi} \circ \sigma\colon S \to \Delta$ and call this procedure Kulikov process of type I started from $P_1, \ldots, P_k$ (or I-process started from $P_1, \ldots, P_k$).

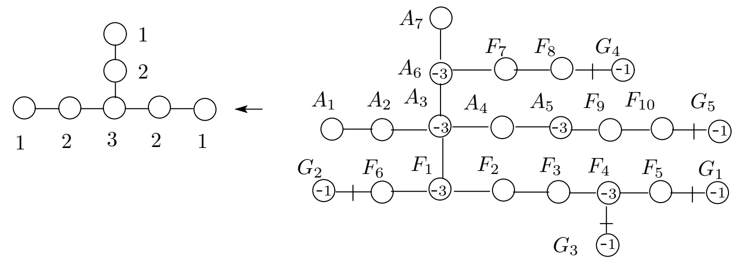
(ii) In I-process of (i), if a component $\overline{A}_{k_j}$ of $\mathrm{supp}(\overline{S}_o)$ contains $P_j^{(1)}$ $(j = 1, \ldots, t_1)$ and $A_{k_j} = \sigma_*^{-1}(\overline{A}_{k_j})$ (*i.e.*, the strict transform of $\overline{A}_{k_j}$ by $\sigma$), then we call $A_{k_j}$ *a root component* of this I-process. Let $B_1, \ldots, B_{t_1}$ be connected components of $B := \mathrm{supp}(S_o) \smallsetminus \mathrm{supp}(\sigma_*^{-1}(\overline{S}_o)$. Each $B_j$ $(j = 1, \ldots, t_1)$ is constructed from all components which are infinitesimally near to $P_j^{(1)}$. We call such $B_j$ *a branch* of $\mathrm{supp}(S_o)$ by this I-process.

(iii) For each branch $B_j$ $(j = 1, \ldots, t_1)$, we denote a partial order between all irreducible components of $B_j$ and the root component. First we denote $A_{k_j} = \sigma_*^{-1}(\overline{A}_{k_j}) \succ F_{j_1}^{(1)} := (\sigma_2 \circ \cdots \circ \sigma_m)_*^{-1}(\sigma_1^{-1}(P_{j_1}^{(1)}))$ where $P_{j_1}^{(1)} \in \overline{A}_{k_j}$. Second, we denote $F_{j_1}^{(1)} \succ F_{j_2}^{(2)} := (\sigma_3 \circ \cdots \circ \sigma_m)_*^{-1}(\sigma_2^{-1}(P_{j_2}^{(2)}))$ if $P_{j_2}^{(2)} \in \sigma_1^{-1}(P_{j_1}^{(1)})$. We continue this for $\sigma_3, \ldots, \sigma_{m-1}$ and $\sigma_m$.

(iv) For any component $F_j^{(i)}$ of a branch $B_j$, let $\ell(F_j^{(i)})$ be the number of blow-ups to produce $F_j^{(i)}$ from the root component $A_j$, and we call it *the length of* $F_j^{(i)}$. Also we define $\ell(A_k) = 0$ for any component $A_k$ of the strict transform of $\operatorname{supp}(S_o)$ through $\sigma$. Further, let $c_R(F_j^{(i)}) = \operatorname{Coeff}_{A_{k_j}} S_o$ (*i.e.*, coefficient of the root of $F_j^{(i)}$) if $A_{k_j}$ is the root of $F_j^{(i)}$.

We explain these terminologies and the situation through the following example:

(2.1)




where $F_1, \ldots, F_{10}, G_1, \ldots, G_5$ are produced through I-process. There are three branches whose root components are $A_3$, $A_5$ and $A_6$. The order between them are given as follows: $A_3 \succ F_1 \succ F_2 \succ F_3 \succ F_4 \succ F_5 \succ G_1, F_1 \succ F_6 \succ G_2, F_4 \succ G_3, A_6 \succ F_7 \succ F_8 \succ G_4$ and $A_5 \succ F_9 \succ F_{10} \succ G_5$. Also we have $\ell(F_1) = 1, \ell(F_8) = 2, \ell(G_1) = 6$ and $\ell(G_3) = 5$.

***Definition 2.2***. — Let $\overline{\Phi}\colon \overline{S} \to \Delta$ be a non-multiple pencil of curves and $Q_1, \ldots, Q_\ell$ non-singular points in $\overline{S}_o$. Namely, they are contained in reduced components (*i.e.*, the coefficient of $S_o$ on the component equals one) and non-singular points of $\operatorname{supp}(\overline{S}_o)$. For each point $Q_j$ ($j = 1, \ldots, \ell$), let's blow-up $s_j$ times at same point $Q_j$, where $s_j \geqslant 2$ for any $i$. Let $\overline{S} \xleftarrow{\psi} S$ be a birational map obtained by these blow-ups. If $Q_j \in \overline{A}_{j_1}$, then any connected component of $\operatorname{supp}(S_o) \setminus \operatorname{supp}(\psi_*^{-1}(\overline{S}_o))$ is a $\mathbb{P}^1$-chain of type $(1, 2, \ldots, 2)$ started from $A_{j_1} = \sigma_*^{-1}(\overline{A}_{j_1})$. We call this *Kulikov process of type* II started from $Q_1, \ldots, Q_k$ (or *II-process* started from $Q_1, \ldots, Q_\ell$).

***Definition 2.3***. — Let $\overline{\overline{\Phi}}\colon \overline{\overline{S}} \to \Delta$ be a non-multiple pencil of curves without any $(-1)$-edge curve. Let $P_1, \ldots, P_k$ (resp. $Q_1, \ldots, Q_\ell$) be non-singular points of $S_o$ (resp. $\operatorname{red}(S_o)$), and assume they are different $k + \ell$ points. Let $\overline{\overline{S}} \xleftarrow{\overline{\overline{\sigma}}} \overline{S}$ be a birational map given by I-processes started from $P_1, \ldots, P_k$, and let $\overline{S} \xleftarrow{\overline{\sigma}} S$ be a birational map given by II-processes started from $Q_1, \ldots, Q_\ell$. We put $\sigma = \overline{\overline{\sigma}} \circ \overline{\sigma}$. Let $A$ be the union of all components of the strict transform of $\operatorname{supp}(\overline{\overline{S}}_o)$ by $\sigma$, and let $F$ be the union of all components in branches by the I-process except for $(-1)$-edge curves. Let $\widetilde{X}$ be a small neighborhood of $A \cup F$ and let $(X, o)$ be a normal surface singularity obtained by contracting $A \cup F$ in $\widetilde{X}$. We call such $(X, o)$ *a singularity obtained from Kulikov-process.*