Séminaires & Congrès
10, 2005, p. 341–350

ADE SURFACE SINGULARITIES, CHAMBERS AND TORIC VARIETIES

by

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Abstract. — We study the link between the positive divisors supported on the exceptional divisor of the minimal resolution of a rational double point and the root systems of Dynkin diagrams. Then, we calculate the toric variety corresponding to the fundamental Weyl chamber.

Résumé **(Singularités ADE des surfaces, chambres et variétés toriques).** — Nous étudions le lien entre les diviseurs positifs à support sur le diviseur exceptionnel de la résolution minimale d'un point double rationnel et les systèmes de racine des diagrammes de Dynkin. Puis, nous calculons la variété torique correspondant à la chambre fondamentale de Weyl.

1. Introduction

A singularity of a normal analytic surface is rational if the geometric genus of the surface doesn't change by a resolution of the singularity. These singularities are rather simple among surface singularities since they are absolutely isolated and their resolutions have some nice combinatoric properties. A classification of rational singularities is done by the dual graph of the minimal resolution according to their multiplicities (see [**11**] for details and related references).

First, DuVal observed that the dual graph of the minimal resolution of a rational singularity of multiplicity 2, called rational double point, with algebraically closed field is one of the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8, briefly ADE diagrams (see [**2**] or [**4**]). This means that the intersection matrix associated to the dual graph of the minimal resolution of a rational double point is the same as the Cartan matrix of the corresponding Dynkin diagram.

2000 ***Mathematics Subject Classification.*** — 32S45, 17B20, 13A50, 14M25.
Key words and phrases. — Rational double singularity, resolution, root system, toric varieties.

The negative definiteness of the intersection matrix of the exceptional divisor of a resolution of a normal surface singularity permits us to study on a set of certain positive divisors supported on the exceptional divisor, which will be called the semigroup of Lipman. By using this set, we can associate a toric variety with a weighted graph whose intersection matrix is negative definite (see [**1**]).

In this work, motivated by a question appeared in [**9**], we give a geometric construction of the roots of an ADE diagram, listed in [**3**] (see Planche I,IV,V,VI,VII). Following [**14**], we observe that the semigroup of Lipman associated with an ADE diagram is the same as the fundamental Weyl chamber of the corresponding root system. In the last section, using [**1**], we describe the toric variety corresponding to the fundamental Weyl chamber of an ADE diagram (see [**1**], [**15**]).

2. Rational Singularities

Let S be a germ at ξ of a complex two dimensional normal space with a singularity at ξ. A *resolution* of S is a complex nonsingular surface with a proper map $\pi : X \to S$ such that its restriction to $X-\pi^{-1}(\xi)$ is an isomorphism and $X-\pi^{-1}(\xi)$ is dense in X. A resolution $\pi : X \to S$ is called *minimal resolution* if any other resolution $\pi' : X' \to S$ factorizes by π. It is well known that the exceptional divisor $E = \pi^{-1}(\xi)$ of π is connected and of dimension 1 (see [**7**], theorem V.5.2). Let us denote by $E_1, \ldots, E_n$ the irreducible components of E. The *intersection matrix* $M(E)$ associated with E is defined by the intersection $(E_i \cdot E_j)$ of the components E_i and E_j, which is the intersection number of E_i and E_j if $i \neq j$, and the first Chern class of the normal bundle to E_i if $i = j$. It is a negative definite matrix (see [**13**]).

Let $\mathcal{G}$ denote the free abelian group generated by the irreducible components of E:

$$\mathcal{G} = \big\{ \textstyle\sum_{i=1}^{n} m_i E_i,\ m_i \in \mathbb{Z} \big\}.$$

The elements of $\mathcal{G}$ are called the *divisors supported on* E. The support of a divisor $Y = \sum_i m_i E_i$ is the set of the components for which $m_i \neq 0$. In the free abelian group $\mathcal{G}$, the intersection matrix $M(E)$ defines a symmetric bilinear form. We shall denote $(Y \cdot Z)$ the value of this bilinear form on a pair (Y, Z) of elements in $\mathcal{G}$. An element of $\mathcal{G}$ in which all the coefficients are non-negative and at least one is positive, is called a *positive divisor*.

***Theorem 2.1* (see [2])**. — *The singularity ξ of S is a rational singularity if and only if the arithmetic genus $\frac{1}{2}(Y \cdot Y + \sum_{i=1}^{n} m_i(w_i - 2)) + 1$ of each positive divisor $Y = \sum_{i=1}^{n} m_i E_i$ in $\mathcal{G}$ is $\leqslant 0$ where $w_i = -(E_i \cdot E_i)$.*

Assume that $\pi : X \to S$ is a resolution of a normal surface singularity which is not necessarily rational. Let f be an element of the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{S,\xi}$. Then the divisor $(\pi^* f)$ of f on X is written as $(\pi^* f) = Y + T_f$ where Y is a positive divisor supported on the exceptional divisor E of π and T_f, called the strict transform of f

by π, intersects E in finitely many point at most. Since $((\pi^* f) \cdot E_i) = 0$ for all i, we obtain $(Y \cdot E_i) \leqslant 0$ for all i. The inverse is true when the singularity ξ is rational. We mean that, if Y is a positive divisor on X such that $(Y \cdot E_i) = -(T \cdot E_i)$ for all i, then there exists a function f in $\mathcal{M}$ such that $(\pi^* f) = Y + T$ (see [**2**]). Now, as in [**12**] (see section 18), let us consider the set

$$\mathcal{E}^+(E) = \{Y \in \mathcal{G} \mid (Y \cdot E_i) \leqslant 0 \text{ for all } i\}$$

By [**18**], this set is not empty. It is an additive semigroup: For $Y_1, Y_2 \in \mathcal{E}^+(E)$, we have $Y_1 + Y_2 \in \mathcal{E}^+(E)$.

Definition 2.2. — The set $\mathcal{E}^+(E)$ is called the semigroup of Lipman.

Since E is connected, for all $Y = \sum m_i E_i$ in $\mathcal{E}^+(E)$, we have $m_i \geqslant 1$ for all i. A partial order on $\mathcal{E}^+(E)$ is defined as follows: For two elements $Y_1 = \sum_{i=1}^n a_i E_i$ and $Y_2 = \sum_{i=1}^n b_i E_i$ of $\mathcal{E}^+(E)$, we say $Y_1 \leqslant Y_2$ if $a_i \leqslant b_i$ for all i. The smallest element of this set is called *the fundamental cycle* of the resolution π. The proposition 4.1 in [**10**], gives the following algorithm to construct the fundamental cycle of a given E:

Let us denote by Z the fundamental cycle of π. Consider $Z_1 = \sum_{i=1}^n E_i$. If $(Z_1 \cdot E_i) \leqslant 0$ for all i, then $Z_1 = Z$. If else, there exists an E_{i_1} such that $(Z_1 \cdot E_{i_1}) > 0$; in this case, we put $Z_2 = Z_1 + E_{i_1}$ and we see whether $(Z_2 \cdot E_i) \leqslant 0$ for all i. The term Z_j, $(j \geqslant 1)$, of the sequence satisfies, either $(Z_j \cdot E_i) \leqslant 0$ for all i, then we put $Z = Z_j$, or there is an irreducible component E_{i_j} such that $(Z_j \cdot E_{i_j}) > 0$, then we put $Z_{j+1} = Z_j + E_{i_j}$. Thus the fundamental cycle of π is the first cycle Z_k of this sequence such that $(Z_k \cdot E_i) \leqslant 0$ for all i. By the same method, we can construct all other elements of $\mathcal{E}^+(E)$ (see [**14**] or [**17**]).

The following result of Artin characterize what an exceptional divisor of a resolution of a rational singularity looks like:

***Theorem 2.3* (see [2])**. — *A singularity of a normal analytic surface in $\mathbb{C}^N$ is rational if and only if the arithmetic genus of the fundamental cycle of the exceptional divisor of a resolution of the singularity vanishes.*

This gives:

***Corollary 2.4* (see [2])**. — *The exceptional divisor of any resolution of a rational singularity is normal crossing, with each E_i nonsingular and of genus zero, and any two distinct components intersect transversally at most in one point.*

A proof of this corollary can be found also in [**17**].

Then the *dual graph* associated with the exceptional divisor of a resolution of a rational singularity, in which each E_i is represented by a vertex and each intersection point is represented by an edge between the vertices corresponding to the intersecting components, is a tree. Each vertex in the dual graph is weighted by $-(E_i \cdot E_i)$. Conversely, with a given weighted graph, by plumbing, we can associate a *configuration* *of curves* embedded in a nonsingular surface and, if such a configuration of curves satisfies theorem 2.3, its contraction gives a rational singularity of a normal analytic surface (see [**6**], [**11**]).

Example 2.5. — A configuration of curves associated with an ADE diagram is contracted to a rational singularity of a normal analytic surface.

Moreover, we have:

***Proposition 2.6* (see [2])**. — *Let $\pi : X \to S$ be the minimal resolution of the rational singularity ξ of S. Then the multiplicity of S at ξ equals $-(Z \cdot Z)$ where Z is the fundamental cycle of π.*

Recall that the minimal resolution is characterized by $(E_i \cdot E_i) \leqslant -2$ for all irreducible components E_i of the exceptional divisor. A rational double point is a rational singularity for which the fundamental cycle of the minimal resolution satisfies $(Z \cdot Z) = -2$. We know that a rational double point of a surface is defined by the power series with the form $f(x, y) + z^2 = 0$. By using the results given above, we deduce:

***Proposition 2.7* (see [2] or [4])**. — *A normal analytic surface singularity is a rational double point if and only if the exceptional divisor of the minimal resolution of the singularity is a configuration of curves associated with one of the ADE diagrams.*

3. Root systems of rational double points

There is a well known construction of ADE diagrams starting from a semisimple Lie algebra. In this section, we are interested in the inverse of that construction, as suggested in [**9**]. We will see that, using the geometry of a Dynkin diagram, we can obtain the roots of the corresponding semisimple Lie algebra. This gives a partial answer to the question of Ito and Nakamura (see [**9**], p. 194).

Let V be an euclidean space endowed with a positive definite symmetric bilinear form $(,)$. A reflection s on V is an orthogonal transformation $s : V \to V$ such that, for $v \in V$, $s(v) = -v$ and it fixes pointwise the hyperplane $H_v = \{u \in V \mid (u, v) = 0\}$ of V. We can describe the reflection by the formula $s_v(u) = u - \frac{2(u,v)}{(v,v)} v$.

Definition 3.1. — A subset R of V is called a root system if

(i) it is finite, generates V and doesn't contain 0,
(ii) for every $v \in R$, there exists a unique reflection s_v such that $s_v(R) = R$,
(iii) for every $v \in R$, the only multiples of v in R are $\pm v$,
(iv) for $u, v \in R$, we have $\frac{2(u,v)}{(v,v)} \in \mathbb{Z}$.

The finite group generated by the reflections is called the Weyl group.
See [**8**] for more details.

In what follows, E will denote a configuration of curves associated with an ADE diagram, called ADE configuration. Now, following [**14**], (see p. 158), we want to establish the relation between the root systems and the semigroup of Lipman of E. Denote by $E_1, \ldots, E_n$ the irreducible components of E. We know that $(E_i \cdot E_j)$ equals -2 if $i = j$ and equals 0 or 1 if $i \neq j$ (see [**4**] or [**12**]). Now, consider the following subset of $\mathcal{G}$:

$$R(E) = \{Y \in \mathcal{G} \mid (Y \cdot Y) = -2\}.$$

***Proposition 3.2* (see [14])**. — *The set $R(E)$ is a root system.*

Replacing the inner product in the definition above by the symmetric bilinear form defined by the intersection matrix $M(E)$, we can see that $R(E)$ satisfies the conditions of the definition above.

We will call *root divisors* the elements of $R(E)$. By definition, $E_1, \ldots, E_n$ and $-E_1, \cdots, -E_n$ are root divisors but $E_i - E_j$ is not a root divisor since $(E_i - E_j \cdot E_i - E_j) \neq -2$ for any $i \neq j$. Let us denote $B = \{E_1, \ldots, E_n\}$. We can see that B is a vector space basis of $R(E)$ in $\mathcal{G} \otimes_{\mathbb{Z}} \mathbb{R}$ and every element Y in $R(E)$ can be written as the sum of E_i's with coefficients all nonnegative or all nonpositive (compare with [**8**], pp. 47-48). If we denote by $R^+(E)$ the set of the elements of $R(E)$ with coefficients all nonnegative, then we have $R(E) = R^+(E) \cup (-R^+(E))$.

***Proposition 3.3* (see [14])**. — *Let $Z = \sum_{i=1}^n a_i E_i$ be the fundamental cycle of E. Then, for each root divisor $Y = \sum_{i=1}^n m_i E_i$ in $R(E)$, we have $m_1 \leqslant a_1, \ldots, m_n \leqslant a_n$.*

The fundamental cycle is called the highest (or biggest) root divisor in $R(E)$.

Proof. — Since E is the exceptional divisor of the minimal resolution of a rational double point, we have $(Z \cdot Z) = -2$. So $Z \in R(E)$. Assume that there is a positive divisor Y in $R(E)$ such that $Y > Z$ and $(Y \cdot Y) = -2$. So we have $Y = Z + D$ where D is a positive divisor. This gives $(Y \cdot Y) = (Z \cdot Z) + 2(Z \cdot D) + (D \cdot D)$. Thus $2(Z \cdot D) = -(D \cdot D)$. Since Z is the fundamental cycle, we have $(Z \cdot E_i) \leqslant 0$ for all i, so $(Z \cdot D) \leqslant 0$. This implies $D = 0$. □

Hence, we can calculate the highest root divisor by the algorithm of Laufer given in the preceding section. The following proposition gives an algorithm to construct all elements of $R(E)$ from Z by using B:

Theorem 3.4. — *Let $R^+(E) = \{Y_0, \ldots, Y_k\}$ with $Y_k = Z$. Then, for each $j = 0, \ldots, k-1$, there exists an element Y_t in $R^+(E)$ such that $(Y_t \cdot E_i) = k_i < 0$ and $Y_j = Y_t + k_i E_i$ for some i. Inversely, for each E_i in B such that $(Y_t \cdot E_i) = k_i < 0$, $Y_t + k_i E_i$ is a root divisor in $R(E)$.*

Proof. — The existence of at least one irreducible component E_i in each Y_j such that $(Y_j \cdot E_i) < 0$ is due to negative definiteness of the intersection matrix. Then, theorem follows from the fact that $(Y_t + k_i E_i) \cdot (Y_t + k_i E_i) = -2$. □